Equations in curvilinear coordinates for fluids with non-constant viscosity

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Abstract
Explicit formulae for the viscosity stress tensor, equations of motion and kinetic energy dissipation are provided in cylindrical and spherical coordinates for Newtonian fluids with non-constant viscosity.

Keywords: Newtonian fluids, non-constant viscosity.

Resumen
Las formulas explícitas para el tensor, ecuaciones de movimiento y la disipación de la energía cinética son proporcionadas en coordenadas cilíndricas y esféricas para fluidos Newtonianos con viscosidad no-constante.

Palabras clave: Fluidos Newtonianos, viscosidad no-constante.

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I. INTRODUCTION

The coefficients of viscosity η and ς in a viscid fluid vary with temperature and pressure. In general *T* and *p*, and therefore η and ς, are not constant throughout the fluid. Taking this fact into account adds new terms to the equations of motion and to the formula of kinetic energy dissipation. As far as we know, these formulae in cylindrical and spherical coordinates have appeared neither in the specialized literature nor in physics manuals. The reason for this omission is due to the fact that in most cases where fluid mechanics is applied, the hypothesis of constant viscosity is a reasonable assumption. But in large extended physical systems such as planets and stars it might be that this hypothesis is no longer valid. Thus, the purpose of this paper is the deduction and presentation of these formulae, in curvilinear coordinates, for fluids when viscosity is not constant.

On the other hand, the main academic interest of this paper lies on the fact that students are usually familiarized with tensors in different coordinate basis but they rarely have dealt with the unitary vectors of these basis. In this paper, they have the opportunity to work with these unitary vectors in curvilinear coordinates regarding both their coordinate transformations and their derivatives.

The motivation for this work arose from Membrado and Pacheco [1], hereafter *MP*. There we studied the rotation of the atmosphere under the simplifying hypothesis that molecular viscosity is solely responsible for the differential rotation of the successive air layers. In *MP* we used the formalism of classical fluids mechanics and, as we had to deal with a fluid where viscosity is not a constant, and in curvilinear coordinates, we needed the above mentioned equations though in a restricted form. This paper is organized as follows. In Section 2, after mentioning several generalities about fluids, we write the viscosity stress tensor and the equations of motion of a viscous fluid. In Section 3, the components of the viscosity stress tensor and those of the equations of motion are deduced in cylindrical and spherical coordinates. In Section 4, we present the general equation of the dissipation of kinetic energy in viscous fluids and its form in curvilinear coordinates. As an example, this equation is utilized in Section 5 to compute the dissipation of kinetic energy in the solution found in *MP* for the atmospheric rotation. Finally, in Section 6, we set out the conclusions.

II. GENERALITIES ABOUT FLUIDS AND VISCOSITY STRESS TENSOR

The motion of any fluid is subject to a kinematic constraint based upon the conservation of mass. This is expressed by means of the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{v}) = -\rho\vec{\nabla}\cdot\vec{v} - \vec{v}\cdot\vec{\nabla}\rho\,, \tag{1}$$

where $\vec{v}$ is the velocity of the fluid and ρ the mass density. When ρ is constant, *i.e.,* the fluid is incompressible, Eq. (1) is simplified to:

$$\vec{\nabla}\cdot\vec{v}=0. \qquad (2)$$

As is well known, in inviscid (ideal) fluids, the *ith* component of the amount of momentum flowing in unit time through the unit area perpendicular to the x_k axis is given, in tensorial form, by:

$$\Pi_{ik}=p\delta_{ik}+\rho v_i v_k. \qquad (3)$$

This tensor Π_{ik} is called the momentum flux density tensor and p is pressure. The equation of motion for an ideal fluid is known as the Euler equation and reads as follows (henceforth, summation over repeated suffixes is assumed):

$$\frac{\partial \rho v_i}{\partial t}=-\frac{\partial \Pi_{ik}}{\partial x_k} \qquad (4)$$

or

$$\rho\left[\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right]=-\frac{\partial p}{\partial x_i}. \qquad (5)$$

In viscous fluids, the tensor written in (3) is generalized by adding a new term which is responsible for the viscous transfer of momentum in the fluid:

$$\Pi_{ik}=-p\delta_{ik}+\rho v_i v_k-\sigma'_{ik}=-\sigma_{ik}+\rho v_i v_k; \qquad (6)$$

the tensor σ_{ik} is called the stress tensor and σ'_{ik} the viscosity stress tensor. Thus, they are related by

$$\sigma_{ik}=\sigma'_{ik}-p\delta_{ik}. \qquad (7)$$

Using general arguments (see for example, Landau and Lifshitz [2]), the σ'_{ik} tensor for a Newtonian fluid is expressed by:

$$\sigma'_{ik}=\eta\left[\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right]+\varsigma\delta_{ik}\frac{\partial v_l}{\partial x_l}, \qquad (8)$$

which is symmetric. The quantities η and ς are called coefficients of viscosity and are both positive.

Then the equations of motion of a viscous fluid, in the most general form, are obtained by adding $\partial\sigma'_{ik}/\partial x_k$ to the right hand side of the Euler equation, given in (5). The result is:

$$\rho\left[\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right]=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_i}\left[\varsigma\frac{\partial v_l}{\partial x_l}\right]+\frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)\right], \qquad (9)$$

which is equivalent to

$$\rho\frac{\partial\vec{v}}{\partial t}+\rho(\vec{v}\cdot\vec{\nabla})\vec{v}=-\vec{\nabla}p+\vec{\nabla}\cdot\vec{\vec{\sigma}}'-\rho\vec{\nabla}\Phi, \qquad (10)$$

where $\vec{\vec{\sigma}}'$ is the notation for the intrinsic form of the viscosity stress tensor. In the added term, Φ represents an external field acting on the fluid, typically a gravitational field.

This equation without Φ, for an incompressible fluid and for η and ς constants, becomes

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{1}{\rho}\vec{\nabla}p+\frac{\eta}{\rho}\nabla^2\vec{v}, \qquad (11)$$

which is the Navier-Stokes equation.

III. VISCOSITY STRESS TENSOR AND FLUID EQUATIONS IN CYLINDRICAL AND SPHERICAL COORDINATES

The components of σ'_{ik} in Eq. (8) correspond to Cartesian coordinates. To deduce these components in cylindrical or spherical coordinates, we start by writing this tensor in its intrinsic form (see for example, McQuarrie [3]), namely:

$$\vec{\vec{\sigma}}'=\eta[(\vec{\nabla}\otimes\vec{v})+(\vec{\nabla}\otimes\vec{v})^T]+(\zeta-\frac{2}{3}\eta)(\vec{\nabla}\cdot\vec{v})\vec{\vec{I}}, \qquad (12)$$

where $\vec{\vec{I}}$ is the unity tensor, the symbol $\vec{a}\otimes\vec{b}$ represent the dyadic product of the two vectors and the superscript T means transposition of the operator; *i.e.*, $(\vec{a}\otimes\vec{b})^T=\vec{b}\otimes\vec{a}$.

A. Cylindrical coordinates (r, θ, z)

In these coordinates, the gradient operator and the velocity vector have the form:

$$\vec{\nabla}=\hat{r}\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}+\hat{z}\frac{\partial}{\partial z}, \qquad (13a)$$

$$\vec{v}=\hat{r}v_r+\hat{\theta}v_\theta+\hat{z}v_z. \qquad (13b)$$

Then, to compute (12) one has to bear in mind that the three unitary vectors are orthogonal, and that the derivative operators act not only on the components of the velocity but also on the unitary vectors:

$$\frac{\partial\hat{r}}{\partial\theta}=\hat{\theta}, \qquad (14a)$$

$$\frac{\partial\hat{\theta}}{\partial\theta}=-\hat{r}. \qquad (14b)$$

After performing the derivation of the vectors and comparing the result with

$$\vec{\vec{\sigma}}'=\sigma'_{rr}\,\hat{r}\otimes\hat{r}+\sigma'_{r\theta}\,\hat{r}\otimes\hat{\theta}+\sigma'_{rz}\,\hat{r}\otimes\hat{z}$$
$$+\sigma'_{\theta r}\,\hat{\theta}\otimes\hat{r}+\sigma'_{\theta\theta}\,\hat{\theta}\otimes\hat{\theta}+\sigma'_{\theta z}\,\hat{\theta}\otimes\hat{z} \qquad (15)$$
$$+\sigma'_{zr}\,\hat{z}\otimes\hat{r}+\sigma'_{z\theta}\,\hat{z}\otimes\hat{\theta}+\sigma'_{zz}\,\hat{z}\otimes\hat{z}\,,$$

the nine components of the tensor are identified. They are:

$$\sigma'_{rr}=\eta\left[2\frac{\partial v_r}{\partial r}\right]+\left[\varsigma-\frac{2}{3}\eta\right]\left[\vec{\nabla}\cdot\vec{v}\right], \qquad (16a)$$

$$\sigma'_{r\theta}=\eta\left[\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right]=\sigma'_{\theta r}\,, \qquad (16b)$$

$$\sigma'_{rz}=\eta\left[\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right]=\sigma'_{zr}\,, \qquad (16c)$$

$$\sigma'_{\theta\theta}=\eta\left[\frac{2}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{2v_r}{r}\right]+\left[\varsigma-\frac{2}{3}\eta\right]\left[\vec{\nabla}\cdot\vec{v}\right], \qquad (16d)$$

$$\sigma'_{\theta z}=\eta\left[\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right]=\sigma'_{z\theta}\,, \qquad (16e)$$

$$\sigma'_{zz}=\eta\left[2\frac{\partial v_z}{\partial z}\right]+\left[\varsigma-\frac{2}{3}\eta\right]\left[\vec{\nabla}\cdot\vec{v}\right], \qquad (16f)$$

where

$$\vec{\nabla}\cdot\vec{v}=\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}+\frac{v_r}{r}. \qquad (17)$$

The form of the continuity equation in these coordinates is

$$\frac{\partial\rho}{\partial t}=-\left[\frac{\partial(\rho v_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta}+\frac{\partial(\rho v_z)}{\partial z}+\frac{\rho v_r}{r}\right]. \qquad (18)$$

Now, as is expressed in (10), the equation of motion requires the computation of the divergence of the tensor $\vec{\vec{\sigma}}'$. As said above, the derivatives act also on the unitary vectors. Performing this derivation first and then grouping terms, one finds:

$$\vec{\nabla}\cdot\vec{\vec{\sigma}}'=\hat{r}\left[\frac{\partial\sigma'_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma'_{\theta r}}{\partial\theta}+\frac{\partial\sigma'_{zr}}{\partial z}+\frac{1}{r}(\sigma'_{rr}-\sigma'_{\theta\theta})\right]$$
$$+\hat{\theta}\left[\frac{\partial\sigma'_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma'_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma'_{z\theta}}{\partial z}+\frac{1}{r}(\sigma'_{r\theta}+\sigma'_{\theta r})\right] \qquad (19)$$
$$+\hat{z}\left[\frac{\partial\sigma'_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\sigma'_{\theta z}}{\partial\theta}+\frac{\partial\sigma'_{zz}}{\partial z}+\frac{1}{r}\sigma'_{rz}\right].$$

To obtain (19), (14) has been used together with the identity

$$\hat{a}\cdot(\hat{b}\otimes\hat{c})=(\hat{a}\cdot\hat{b})\cdot\hat{c}\,. \qquad (20)$$

Finally, performing the derivation of the tensor components given in (16), the three components of the equation of motion are:

r component:

$$\rho\frac{\partial v_r}{\partial t}+\rho\left[v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right]=$$
$$-\frac{\partial p}{\partial r}-\rho\frac{\partial\Phi}{\partial r}+\frac{\partial\eta}{\partial r}\left[2\frac{\partial v_r}{\partial r}-\vec{\nabla}\cdot\vec{v}\right]$$
$$+\frac{1}{r}\frac{\partial\eta}{\partial\theta}\left[\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right] \qquad (21)$$
$$+\frac{\partial\eta}{\partial z}\left[\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right]+\frac{\partial}{\partial r}\left[\left(\zeta+\frac{\eta}{3}\right)\left(\vec{\nabla}\cdot\vec{v}\right)\right]$$
$$+\eta\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{v_r}{r^2}\right];$$

θ component:

$$\rho\frac{\partial v_\theta}{\partial t}+\rho\left[v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right]=$$
$$-\frac{1}{r}\frac{\partial p}{\partial\theta}-\frac{\rho}{r}\frac{\partial\Phi}{\partial\theta}+\frac{\partial\eta}{\partial r}\left[\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right]$$
$$+\frac{1}{r}\frac{\partial\eta}{\partial\theta}\left[\frac{2}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{2v_r}{r}-\vec{\nabla}\cdot\vec{v}\right] \qquad (22)$$
$$+\frac{\partial\eta}{\partial z}\left[\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(\zeta+\frac{\eta}{3}\right)\left(\vec{\nabla}\cdot\vec{v}\right)\right]$$
$$+\eta\left[\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}\right];$$

z component:

$$\rho\frac{\partial v_z}{\partial t}+\rho\left[v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right]=$$
$$-\frac{\partial p}{\partial z}-\rho\frac{\partial\Phi}{\partial z}+\frac{\partial\eta}{\partial r}\left[\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right]$$
$$+\frac{1}{r}\frac{\partial\eta}{\partial\theta}\left[\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right] \qquad (23)$$
$$+\frac{\partial\eta}{\partial z}\left[2\frac{\partial v_z}{\partial z}-\vec{\nabla}\cdot\vec{v}\right]+\frac{\partial}{\partial z}\left[\left(\zeta+\frac{\eta}{3}\right)\left(\vec{\nabla}\cdot\vec{v}\right)\right]$$
$$+\eta\left[\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right].$$

Note the presence of terms involving derivatives of the viscosity. These terms are absent in fluid mechanics books and physics vade mecums.

B. Spherical coordinates (*R*, ϕ, θ)

The calculus in spherical coordinates (R,ϕ,θ), where ϕ and θ are the zenith and the azimuth angle respectively, is similar to that explained above for cylindrical coordinates.

Here, the gradient operator and the velocity are

$$\vec{\nabla}=\hat{R}\frac{\partial}{\partial R}+\frac{\hat{\phi}}{R}\frac{\partial}{\partial\phi}+\frac{\hat{\theta}}{R\sin\phi}\frac{\partial}{\partial\theta}, \qquad (24a)$$

$$\vec{v}=\hat{R}v_R+\hat{\phi}v_\phi+\hat{\theta}v_\theta. \qquad (24b)$$

The derivatives of the unitary vectors are

$$\frac{\partial \hat{R}}{\partial \phi} = \hat{\phi}, \tag{25a}$$

$$\frac{\partial \hat{R}}{\partial \theta} = \hat{\theta}\sin\phi, \tag{25b}$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{R}, \tag{25c}$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = \hat{\theta}\cos\phi, \tag{25d}$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{R}\sin\phi - \hat{\phi}\cos\phi. \tag{25e}$$

And the list of components of the viscosity stress tensor is:

$$\sigma'_{RR} = \eta\left[2\frac{\partial v_R}{\partial R}\right] + \left[\varsigma - \frac{2}{3}\eta\right]\left[\vec{\nabla}\cdot\vec{v}\right], \tag{26a}$$

$$\sigma'_{R\phi} = \eta\left[\frac{\partial v_\phi}{\partial R} + \frac{1}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi}{R}\right] = \sigma'_{\phi R}, \tag{26b}$$

$$\sigma'_{R\theta} = \eta\left[\frac{\partial v_\theta}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta}{R}\right] = \sigma'_{\theta R}, \tag{26c}$$

$$\sigma'_{\phi\phi} = \eta\left[\frac{2}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{2v_R}{R}\right] + \left[\varsigma - \frac{2}{3}\eta\right]\left[\vec{\nabla}\cdot\vec{v}\right], \tag{26d}$$

$$\sigma'_{\phi\theta} = \eta\left[\frac{1}{R}\frac{\partial v_\theta}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\theta\cos\phi}{R\sin\phi}\right] = \sigma'_{\theta\phi}, \tag{26e}$$

$$\begin{aligned}\sigma'_{\theta\theta} = {} & \eta\left[\frac{2}{R\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{2v_R}{R} + \frac{2v_\phi\cos\phi}{R\sin\phi}\right] \\ & + \left[\varsigma - \frac{2}{3}\eta\right]\left[\vec{\nabla}\cdot\vec{v}\right].\end{aligned} \tag{26f}$$

where

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_R}{\partial R} + \frac{1}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{2v_R}{R} + \frac{v_\phi\cos\phi}{R\sin\phi}. \tag{27}$$

The form of the continuity equation in these coordinates is

$$\begin{aligned}\frac{\partial \rho}{\partial t} = -\Bigg[& \frac{\partial(\rho v_R)}{\partial R} + \frac{1}{R}\frac{\partial(\rho v_\phi)}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial(\rho v_\theta)}{\partial \theta} \\ & + \frac{2\rho v_R}{R} + \frac{\rho v_\phi\cos\phi}{R\sin\phi}\Bigg].\end{aligned} \tag{28}$$

After performing the derivation of the unitary vectors, the divergence of the tensor reads as follows:

$$\begin{aligned}\vec{\nabla}\cdot\vec{\vec{\sigma}}' = {} & \hat{R}\Bigg[\frac{\partial \sigma'_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \sigma'_{\phi R}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma'_{\theta R}}{\partial \theta} \\ & + \frac{\cos\phi}{R\sin\phi}\sigma'_{\phi R} + \frac{1}{R}(2\sigma'_{RR} - \sigma'_{\phi\phi} - \sigma'_{\theta\theta})\Bigg] \\ & + \hat{\phi}\Bigg[\frac{\partial \sigma'_{R\phi}}{\partial R} + \frac{1}{R}\frac{\partial \sigma'_{\phi\phi}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma'_{\theta\phi}}{\partial \theta} \\ & + \frac{\cos\phi}{R\sin\phi}(\sigma'_{\phi\phi} - \sigma'_{\theta\theta}) + \frac{1}{R}(2\sigma'_{R\phi} + \sigma'_{\phi R})\Bigg] \\ & + \hat{\theta}\Bigg[\frac{\partial \sigma'_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial \sigma'_{\phi\theta}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma'_{\theta\theta}}{\partial \theta} \\ & + \frac{\cos\phi}{R\sin\phi}(\sigma'_{\phi\theta} + \sigma'_{\theta\phi}) + \frac{1}{R}(2\sigma'_{R\theta} + \sigma'_{\theta R})\Bigg].\end{aligned} \tag{29}$$

And using the tensor components collected in (26), the three components of the equations of motion are:

R component:

$$\begin{aligned}\rho\frac{\partial v_R}{\partial t} + \rho\left[v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} + \frac{v_\theta}{R\sin\phi}\frac{\partial v_R}{\partial \theta} - \frac{(v_\phi^2 + v_\theta^2)}{R}\right] = & \\ -\frac{\partial p}{\partial R} - \rho\frac{\partial \Phi}{\partial R} + \frac{\partial \eta}{\partial R}\left[2\frac{\partial v_R}{\partial R} - \vec{\nabla}\cdot\vec{v}\right] & \\ + \frac{1}{R}\frac{\partial \eta}{\partial \phi}\left[\frac{\partial v_\phi}{\partial R} + \frac{1}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi}{R}\right] & \\ + \frac{1}{R\sin\phi}\frac{\partial \eta}{\partial \theta}\left[\frac{\partial v_\theta}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta}{R}\right] & \\ + \frac{\partial}{\partial R}\left[\left(\zeta + \frac{\eta}{3}\right)\left(\vec{\nabla}\cdot\vec{v}\right)\right] & \\ + \eta\Bigg[\frac{\partial^2 v_R}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 v_R}{\partial \phi^2} + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 v_R}{\partial \theta^2} + \frac{2}{R}\frac{\partial v_R}{\partial R} & \\ + \frac{\cos\phi}{R^2\sin\phi}\frac{\partial v_R}{\partial \phi} - \frac{2}{R^2}\frac{\partial v_\phi}{\partial \phi} - \frac{2}{R^2\sin\phi}\frac{\partial v_\theta}{\partial \theta} & \\ - \frac{2v_R}{R^2} - \frac{2v_\phi\cos\phi}{R^2\sin\phi}\Bigg]; &\end{aligned} \tag{30}$$

ϕ component:

$$\begin{aligned}\rho\frac{\partial v_\phi}{\partial t} + \rho\Bigg[v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\theta}{R\sin\phi}\frac{\partial v_\phi}{\partial \theta} & \\ + \frac{v_R v_\phi}{R} - \frac{v_\theta^2\cos\phi}{R\sin\phi}\Bigg] = & \\ -\frac{1}{R}\frac{\partial p}{\partial \phi} - \frac{\rho}{R}\frac{\partial \Phi}{\partial \phi} + \frac{\partial \eta}{\partial R}\left[\frac{\partial v_\phi}{\partial R} + \frac{1}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi}{R}\right] & \\ + \frac{1}{R}\frac{\partial \eta}{\partial \phi}\left[\frac{2}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{2v_R}{R} - \vec{\nabla}\cdot\vec{v}\right] & \\ + \frac{1}{R\sin\phi}\frac{\partial \eta}{\partial \theta}\left[\frac{1}{R}\frac{\partial v_\theta}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\theta\cos\phi}{R\sin\phi}\right] & \\ + \frac{1}{R}\frac{\partial}{\partial \phi}\left[\left(\zeta + \frac{\eta}{3}\right)\left(\vec{\nabla}\cdot\vec{v}\right)\right] & \\ + \eta\Bigg[\frac{\partial^2 v_\phi}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 v_\phi}{\partial \theta^2} + \frac{2}{R}\frac{\partial v_\phi}{\partial R} &\end{aligned} \tag{31}$$

$$+\frac{2}{R^2}\frac{\partial v_R}{\partial\phi}+\frac{\cos\phi}{R^2\sin\phi}\frac{\partial v_\phi}{\partial\phi}-\frac{2\cos\phi}{R^2\sin^2\phi}\frac{\partial v_\theta}{\partial\theta}-\frac{v_\phi}{R^2\sin^2\phi}\Bigg];$$

θ component:

$$\rho\frac{\partial v_\theta}{\partial t}+\rho\left[v_R\frac{\partial v_\theta}{\partial R}+\frac{v_\varphi}{R}\frac{\partial v_\theta}{\partial\varphi}+\frac{v_\theta}{R\sin\varphi}\frac{\partial v_\theta}{\partial\theta}+\frac{v_R v_\theta}{R}+\frac{v_\phi v_\theta\cos\phi}{R\sin\phi}\right]=$$
$$-\frac{1}{R\sin\phi}\frac{\partial p}{\partial\theta}-\frac{\rho}{R\sin\phi}\frac{\partial\Phi}{\partial\theta}+\frac{\partial\eta}{\partial R}\left[\frac{\partial v_\theta}{\partial R}+\frac{1}{R\sin\phi}\frac{\partial v_R}{\partial\theta}-\frac{v_\theta}{R}\right]$$
$$+\frac{1}{R}\frac{\partial\eta}{\partial\phi}\left[\frac{1}{R}\frac{\partial v_\theta}{\partial\phi}+\frac{1}{R\sin\phi}\frac{\partial v_\phi}{\partial\theta}-\frac{v_\theta\cos\phi}{R\sin\phi}\right]$$
$$+\frac{1}{R\sin\phi}\frac{\partial\eta}{\partial\theta}\left[\frac{2}{R\sin\phi}\frac{\partial v_\theta}{\partial\theta}+\frac{2v_R}{R}+\frac{2v_\phi\cos\phi}{R\sin\phi}-\vec\nabla\cdot\vec v\right] \quad (32)$$
$$+\frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}\left[\left(\zeta+\frac{\eta}{3}\right)\left(\vec\nabla\cdot\vec v\right)\right]$$
$$+\eta\left[\frac{\partial^2 v_\theta}{\partial R^2}+\frac{1}{R^2}\frac{\partial^2 v_\theta}{\partial\phi^2}+\frac{1}{R^2\sin^2\phi}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{R}\frac{\partial v_\theta}{\partial R}\right.$$
$$+\frac{\cos\phi}{R^2\sin\phi}\frac{\partial v_\theta}{\partial\phi}+\frac{2}{R^2\sin\phi}\frac{\partial v_R}{\partial\theta}+\frac{2\cos\phi}{R^2\sin^2\phi}\frac{\partial v_\phi}{\partial\theta}$$
$$\left.-\frac{v_\theta}{R^2\sin^2\phi}\right].$$

IV. EQUATION FOR THE RATE OF ENERGY DISSIPATION IN A FLUID

The kinetic energy in a unit volume and the total kinetic energy in a fluid are equal to

$$\varepsilon_{kin}=\frac{1}{2}\rho v^2\,, \quad (33)$$

and

$$E_{kin}=\int_V\frac{1}{2}\rho v^2\,dV\,, \quad (34)$$

respectively. Assuming that the borders of the volume are fixed, the time variation of the kinetic energy in that volume is

$$\frac{dE_{kin}}{dt}=\int_V\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right)dV\,. \quad (35)$$

Using the equation of motion (10) and the continuity Eq. (1), we find:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right)=-\frac{1}{2}v^2\vec\nabla\cdot(\rho\vec v)-\rho\vec v\cdot\vec\nabla\left(\frac{1}{2}v^2\right)$$
$$-\vec v\cdot\vec\nabla p+\vec v\cdot(\vec\nabla\cdot\vec{\vec\sigma}')-\rho\vec v\cdot\vec\nabla\Phi$$
$$=-\vec\nabla\left(\rho\vec v\frac{1}{2}v^2+p\vec v-\vec v\cdot\vec{\vec\sigma}'+\rho\vec v\Phi\right) \quad (36)$$
$$+p\vec\nabla\cdot\vec v+\Phi\vec\nabla\cdot(\rho\vec v)-\sigma'_{ij}\frac{\partial v_i}{\partial x_j}.$$

This is the most general form to express the kinetic energy dissipation for unit volume in a fluid.

Now, we will express the two terms $\vec v\cdot\vec{\vec\sigma}'$ and $(\vec{\vec\sigma}'\cdot\vec\nabla)\cdot\vec v$ in cylindrical and spherical coordinates.

Cylindrical coordinates:

$$\vec v\cdot\vec{\vec\sigma}'=(v_r\sigma'_{rr}+v_\theta\sigma'_{\theta r}+v_z\sigma'_{zr})\,\hat r$$
$$+(v_r\sigma'_{r\theta}+v_\theta\sigma'_{\theta\theta}+v_z\sigma'_{z\theta})\,\hat\theta \quad (37)$$
$$+(v_r\sigma'_{rz}+v_\theta\sigma'_{\theta z}+v_z\sigma'_{zz})\,\hat z\,;$$

$$(\vec{\vec\sigma}'\cdot\vec\nabla)\cdot\vec v=\eta\left\{2\left[\frac{\partial v_r}{\partial r}\right]^2+\left[\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right]^2\right.$$
$$+\left[\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right]^2+2\left[\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right]^2$$
$$\left.+\left[\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial\theta}\right]^2+2\left[\frac{\partial v_z}{\partial z}\right]^2\right\} \quad (38)$$
$$+\left(\varsigma-\frac{2}{3}\eta\right)(\vec\nabla\cdot\vec v)\left\{\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right\}.$$

Spherical coordinates:

$$\vec v\cdot\vec{\vec\sigma}'=(v_R\sigma'_{RR}+v_\phi\sigma'_{\phi R}+v_\theta\sigma'_{\theta R})\,\hat R$$
$$+(v_R\sigma'_{R\phi}+v_\phi\sigma'_{\phi\phi}+v_\theta\sigma'_{\theta\phi})\,\hat\phi \quad (39)$$
$$+(v_R\sigma'_{R\theta}+v_\phi\sigma'_{\phi\theta}+v_\theta\sigma'_{\theta\theta})\,\hat\theta\,;$$

$$(\vec{\vec\sigma}'\cdot\vec\nabla)\cdot\vec v=\eta\left\{2\left[\frac{\partial v_R}{\partial R}\right]^2+\left[\frac{\partial v_\phi}{\partial R}+\frac{1}{R}\frac{\partial v_R}{\partial\phi}-\frac{v_\phi}{R}\right]^2\right.$$
$$+\left[\frac{\partial v_\theta}{\partial R}+\frac{1}{R\sin\phi}\frac{\partial v_R}{\partial\theta}-\frac{v_\theta}{R}\right]^2+2\left[\frac{1}{R}\frac{\partial v_\phi}{\partial\phi}+\frac{v_R}{R}\right]^2$$
$$+\left[\frac{1}{R}\frac{\partial v_\theta}{\partial\phi}+\frac{1}{R\sin\phi}\frac{\partial v_\phi}{\partial\theta}-\frac{v_\theta\cos\phi}{R\sin\phi}\right]^2$$
$$\left.+2\left[\frac{1}{R\sin\phi}\frac{\partial v_\theta}{\partial\theta}+\frac{v_R}{R}+\frac{v_\phi\cos\phi}{R\sin\phi}\right]^2\right\} \quad (40)$$
$$+\left(\varsigma-\frac{2}{3}\eta\right)(\vec\nabla\cdot\vec v)\left\{\frac{\partial v_R}{\partial R}+\frac{1}{R}\frac{\partial v_\varphi}{\partial\varphi}+2\frac{v_R}{R}\right.$$
$$\left.+\frac{1}{R\sin\varphi}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\varphi\cos\varphi}{R\sin\varphi}\right\}.$$

V. AN EXAMPLE OF KINETIC ENERGY DISSIPATION

As an explicit example of the use of the equation of kinetic energy dissipation, we will apply (36), (37) and (38) to the solution found in *MP for* the atmospheric rotation. In *MP* it was assumed that the vector velocity of the air was as follows:

$$\vec{v} = v_\theta \hat{\theta}\,, \tag{41}$$

where v_θ was a function of (r, z) or equivalently of (r, R) which implies that $\vec{\nabla}\cdot\vec{v} = 0$. This equation is mathematically equivalent to the continuity Eq. (2) of an incompressible fluid, though the atmosphere is definitely not such a fluid.

The explicit solution for the velocity of rotation of the air in *MP* was

$$v(r,R) = \Omega r\left\{\int_R^\infty \frac{dR}{\eta R^4}\right\}\left\{\int_{R_0}^\infty \frac{dR}{\eta R^4}\right\}^{-1}, \tag{42}$$

where Ω and R_0 are the angular velocity and radius of Earth respectively. R is the distance to the center and r the distance to the axis of rotation. As said above, a solution of this type implies that

$$\vec{\nabla}\cdot\vec{v} = 0\,, \tag{43}$$

is fulfilled, and also

$$\vec{\nabla}\cdot(\rho\vec{v}) = 0\,. \tag{44}$$

Equation (44) is a consequence of assuming a steady state. Therefore, in this problem, Eq. (36) is simplified:

$$\frac{dE_{kin}}{dt} = -\int_S \left[\rho\vec{v}\left(\frac{1}{2}\vec{v}^2 + \frac{p}{\rho} + \Phi\right) - \vec{v}\cdot\vec{\vec{\sigma}}'\right]\cdot d\vec{S} - \int_V \sigma'_{ij}\frac{\partial v_i}{\partial x_j}dV\,. \tag{45}$$

The first term in this formula has been obtained using Gauss' theorem.

The volume in these equations can be fixed at will. We will choose a thick spherical shell contained between $R_{\min}$ and $R_{\max}$. Both $R_{\min}$ and $R_{\max}$ are bigger than R_0.

Then, the unitary vector perpendicular to the top surface, in cylindrical coordinates, is

$$\hat{n} = \frac{1}{R}(r,0,z)\,, \tag{46}$$

and using (41), we find

$$\vec{v}\cdot\hat{n} = 0\,, \tag{47}$$

so that

$$(\vec{v}\cdot\vec{\vec{\sigma}}')\cdot\hat{n} = \frac{\eta}{R}\left[r v_\theta\frac{\partial v_\theta}{\partial r} - v_\theta^2 + z v_\theta\frac{\partial v_\theta}{\partial z}\right]. \tag{48}$$

Now, we change the derivatives from (r, z) to (r, R) coordinates and make use of (42). The net result for the surface term in (45) is:

$$\begin{aligned} T_S &= \int_S (\vec{v}\cdot\vec{\vec{\sigma}}')\cdot\hat{n}\,dS \\ &= \frac{8\pi\Omega^2}{3}\left\{-\int_{R_{\max}}^\infty \frac{dR}{\eta R^4} + \int_{R_{\min}}^\infty \frac{dR}{\eta R^4}\right\}\left\{\int_{R_0}^\infty \frac{dR}{\eta R^4}\right\}^{-2} \\ &= \frac{8\pi\Omega^2}{3}\left\{\int_{R_{\min}}^{R_{\max}} \frac{dR}{\eta R^4}\right\}\left\{\int_{R_0}^\infty \frac{dR}{\eta R^4}\right\}^{-2}. \end{aligned} \tag{49}$$

In the upper line of (49) the second term is positive and represents the energy transferred to the air contained in the volume through the bottom surface of the shell. The source of this energy is the rotation of the lower atmospheric layers. The first term is negative and represents the loss of energy of the air in this volume through the top surface. The balance between these terms written in the lower line, T_S, provides the net kinetic energy given to that volume of atmosphere. Let us consider now the volume term in (45). In cylindrical coordinates, the velocity and gradient was given in (13). Using (38) and exploiting the fact that $\vec{\nabla}\cdot\vec{v} = 0$, we find

$$\sigma'_{ij}\frac{\partial v_i}{\partial x_j} = \eta\left[\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2 + \left(\frac{\partial v_\theta}{\partial z}\right)^2\right]. \tag{50}$$

Now, passing from the coordinates (r, z) to (r, R), (50) adopts the form

$$\sigma'_{ij}\frac{\partial v_i}{\partial x_j} = \eta\left(\frac{\partial v_\theta}{\partial R}\right)^2, \tag{51}$$

and using the explicit form of the fluid velocity (42), the result is

$$T_V = -\int_V \sigma'_{ij}\frac{\partial v_i}{\partial x_j}dV = -\frac{8\pi\Omega^2}{3}\left\{\int_{R_{\min}}^{R_{\max}} \frac{dR}{\eta R^4}\right\}\left\{\int_{R_0}^\infty \frac{dR}{\eta R^4}\right\}^{-2}. \tag{52}$$

This is the kinetic energy rate of dissipation in the volume considered. This term, T_V, exactly balances T_S as is required by the principle of energy conservation.

VI. CONCLUSIONS

In this paper we have presented, in the most general form, the viscosity stress tensor and the equations of motion of viscid fluids, for cylindrical and spherical coordinates, with the hypothesis that viscosity is not constant. The complete set of these equations is written in Section 3. The general formula for the rate of dissipation of kinetic energy in these fluids has been also deduced and is written in Section 4. We believe that the set of formulae contained in this paper are original and useful for dealing with Newtonian fluids in extended systems where the symmetry of the problem imposes the use of curvilinear coordinates and where viscosity cannot be considered as a constant.

Some of the formulae collected here are obtained through lengthy but instructive calculations. The details of any of them are available on request.

As an illustrative example, we have computed the kinetic energy dissipation in the solution studied in *MP* for atmospheric rotation. There, the viscosity coefficients were assumed to vary with T but not with p. Besides, with the ansatz assumed for the rotation velocity (41), the formulae are considerably simplified and for example it is possible to recognize that the energy dissipated in a fixed volume of atmosphere is the difference between the energy transferred by the atmosphere through the bottom surface and the energy lost through the top surface.

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