EDVCATIO PHYSICORVM
QVO NON ASCENDAM?

Maxwell-Faraday equation extension to four-dimensions

Adrián G. Cornejo
Electronics and Communications Engineering from Universidad Iberoamericana
Calle Santa Rosa 719, C.P. 76138, Col. Santa Mónica, Querétaro, Querétaro, Mexico.

E-mail: adriang.cornejo@gmail.com

(Received 14 October 2011; accepted 21 December 2011)

Abstract

In this paper, we develop an extension to a four-dimensional frame of reference for the classical Maxwell–Faraday equation by applying the method of directional derivatives.

Keywords: Faraday's law, Maxwell-Faraday equation, Four-dimensions.

Resumen

En este trabajo, desarrollamos una extensión a un marco de referencia de cuatro-dimensiones de la ecuación clásica de Maxwell–Faraday aplicando el método de derivadas direccionales.

Palabras clave: Ley de Faraday, Ecuación de Maxwell-Faraday, Cuatro-dimensiones.

PACS: 02.30.Jr, 03.50.De, 04.40.Nr **ISSN 1870-9095**

I. INTRODUCTION

As background, Faraday's law of the induction [1] (or simply Faraday's law) is based on the Michael Faraday's experiments of 1831 which states that the induced voltage in a closed circuit (for instance, a loop of wire or circuit with electrical current passing through it) is directly proportional to the changing of magnetic flux crossing the surface with the circuit as an edge, which expressed in SI units is given by

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_S \nabla\times\mathbf{E}\cdot d\mathbf{A} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{A}, \qquad (1)$$

where ∇ is the curl (also called rotor or rotationa l, and also denoted as *rot*), $\mathbf{E}$ is the electric field, $d\mathbf{l}$ is the infinitesimal element of contour C, $\mathbf{B}$ is the magnetic field density, S is the arbitrary surface defined by $z = f(x, y)$, whose edge is C (with C and S not necessary stationary) and $d\mathbf{A}$ is the infinitesimal element of area. Direction of electric flux and magnetic flux are given by the right-hand rule. This surface integral can be written in the differential form by applying the Stockes theorem [2, 3], then obtaining the curl in terms of the partial time derivative. Direction of the curl is the axis of rotation and the magnitude of the curl is given by the magnitude of rotation. For the z-axis, its differential form is given by

$$rot_z\mathbf{E} = -\frac{\partial \mathbf{B_z}}{\partial t}. \qquad (2)$$

Thus, magnetic flux changing in time is proportional to the electromotive force. Maxwell included the Faraday's law in his Maxwell's equations [4], known as the Maxwell–Faraday equation. As known, Maxwell's equations are a set of partial differential equations that describes the electromagnetic behavior on a surface or region around an electrical current, unifying the electromagnetism.

On the other hand, four-dimensional space (some times denoted as "4-D") is a dimensional concept derived by generalizing the rules of three-dimensional space [5, 6]. Algebraically it is generated by applying the rules of vectors and coordinate geometry to a space with four dimensions. In particular a vector with four elements (a 4-tuple) can be used to represent a position in four-dimensional space.

In this paper, we develop an extension to four-dimensions for the Maxwell–Faraday equation. Classical mathematical method of directional derivatives is applied to derive the Maxwell–Faraday equation in its differential form, but extended to four-dimensions.

II. MAXWELL-FARADAY EQUATION EXTENSION TO FOUR-DIMENSIONS

Extending the Maxwell's basic model [1, 7] by adding a new dimensional parameter named w, we suppose a surface in a four-dimensional frame of reference given by $w = f(x,$

y, z), where the electric field **E** in four-dimensions is given by the parameters of tetra-vector

$$\mathbf{E} = (\mathbf{E}_w, \mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_\mathbf{z}). \tag{3}$$

In order to determinate the direction of the electrical current through a four-dimensional frame of reference, we suppose an electrical current which follows a 3-dimensional closed trajectory around a central point O as the origin of the coordinate system (for instance, a rectangular circuit in the 3-dimensional space for simplicity, described by the points A, B, C and D). In this way, at the same time such an electrical current also advances positively passing through the points A', B', C' and D' located along the w-axis which is orthogonal to the other three coordinates of the familiar three-dimensional space, where point A and point A' coincide as the same point in a four-dimensions frame of reference, then having components $A(w, x, y, z)$. In the same way, points B and B' coincide as the same point in four-dimensions, and so, C and C', and D and D', respectively, coincide as the same points in four-dimensions. It means that each one of those points is described in a four-dimensional frame of reference. In this way, extending the directional derivatives of a multivariate differentiable scalar function [8, 9] to four-dimensions for a function defined by $w = f(x, y, z)$ along a given unit vector $\mathbf{V}_w = (\mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z)$ at a given point A, represents the instantaneous rate of change of the function, moving through A, in the direction of **V**, yields the function defined by the limit given by

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}\Bigg|_{\substack{y=ctt\\ z=ctt}}, \tag{4}$$

$$\frac{\partial w}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x, y, z)}{\Delta y}\Bigg|_{\substack{x=ctt\\ z=ctt}},$$

$$\frac{\partial w}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x, y, z)}{\Delta z}\Bigg|_{\substack{x=ctt\\ y=ctt}}.$$

Thus, since we are considering an electrical current which enters and leaves respectively in each one of the points of that circuit, respectively, the vector **V** can be considered like the vector of electrical current, where components of vector electric field **E** for each point according to the circulation yield

$$\begin{aligned} \mathbf{E}_A &= [(\mathbf{E}_x - \mathbf{E}_y - \mathbf{E}_z), \\ &\quad (\mathbf{E}_\mathbf{y} - \mathbf{E}_x - \mathbf{E}_z), (\mathbf{E}_z - \mathbf{E}_x - \mathbf{E}_y)], \\ \mathbf{E}_B &= [(\mathbf{E}_x - \mathbf{E}_y - \mathbf{E}_z), (\mathbf{E}_y + \mathbf{E}_x + \mathbf{E}_z), (\mathbf{E}_z - \mathbf{E}_x - \mathbf{E}_y)], \\ \mathbf{E}_C &= [(\mathbf{E}_x + \mathbf{E}_y + \mathbf{E}_z), (\mathbf{E}_y + \mathbf{E}_x + \mathbf{E}_z), (\mathbf{E}_z + \mathbf{E}_x + \mathbf{E}_y)], \\ \mathbf{E}_D &= [(\mathbf{E}_x + \mathbf{E}_y + \mathbf{E}_z), (\mathbf{E}_y - \mathbf{E}_x - \mathbf{E}_z), (\mathbf{E}_z + \mathbf{E}_x + \mathbf{E}_y)]. \end{aligned} \tag{5}$$

In order to determinate the value of these components, it is considered the equation of a straight line in four-dimensions, given by one of the expressions

$$w = \begin{cases} mx + y + z, \\ x + my + z, \\ x + y + mz, \end{cases} \tag{6}$$

where m is the slope of the straight line.

In the case of $w = my + x + z$, slope is given by $m = \Delta x/\Delta y$. Then, we can apply slope to derivate the components of electric field **E** [10], where for y-component yields

$$\mathbf{E}_y = \frac{\partial \mathbf{E}_w}{\partial y} = \frac{d}{dy}\left(\frac{\Delta x}{\Delta y}\mathbf{E}_y + \mathbf{E}_x + \mathbf{E}_z\right)\Bigg|_{\substack{x=ctt\\ z=ctt}}, \tag{7}$$

$$\mathbf{E}_y = \frac{\partial \mathbf{E}_x}{\partial y} y = \frac{\partial \mathbf{E}_x}{\partial y}\left(\frac{\delta y}{2}\right),$$

where δy is the length of segment of circuit along the y-axis, so that from the origin we have the half of such a length as $\delta y/2$. Thus, deriving components for the other points of the circuit, coordinates of electric field **E** for each point when w is a constant, according to the expression (5) are given by

$$\begin{aligned} \mathbf{E}_A = \Bigg[&\left(\mathbf{E}_x - \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y} - \frac{\delta z}{2}\frac{\partial \mathbf{E}_x}{\partial z}\right), \\ &\left(\mathbf{E}_y - \frac{\delta x}{2}\frac{\partial \mathbf{E}_y}{\partial x} - \frac{\delta z}{2}\frac{\partial \mathbf{E}_y}{\partial z}\right), \\ &\left(\mathbf{E}_z - \frac{\delta x}{2}\frac{\partial \mathbf{E}_z}{\partial x} - \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y}\right)\Bigg], \\ \mathbf{E}_B = \Bigg[&\left(\mathbf{E}_x - \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y} - \frac{\delta z}{2}\frac{\partial \mathbf{E}_x}{\partial z}\right), \\ &\left(\mathbf{E}_y + \frac{\delta x}{2}\frac{\partial \mathbf{E}_y}{\partial x} + \frac{\delta z}{2}\frac{\partial \mathbf{E}_y}{\partial z}\right), \\ &\left(\mathbf{E}_z - \frac{\delta x}{2}\frac{\partial \mathbf{E}_z}{\partial x} - \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y}\right)\Bigg], \\ \mathbf{E}_C = \Bigg[&\left(\mathbf{E}_x + \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y} + \frac{\delta z}{2}\frac{\partial \mathbf{E}_x}{\partial z}\right), \\ &\left(\mathbf{E}_y + \frac{\delta x}{2}\frac{\partial \mathbf{E}_y}{\partial x} + \frac{\delta z}{2}\frac{\partial \mathbf{E}_y}{\partial z}\right), \\ &\left(\mathbf{E}_z + \frac{\delta x}{2}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y}\right)\Bigg], \end{aligned} \tag{8}$$

$$\mathbf{E}_D = \left[\left(\mathbf{E}_x + \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y} + \frac{\delta z}{2}\frac{\partial \mathbf{E}_x}{\partial z}\right), \left(\mathbf{E}_y - \frac{\delta x}{2}\frac{\partial \mathbf{E}_y}{\partial x} - \frac{\delta z}{2}\frac{\partial \mathbf{E}_y}{\partial z}\right), \left(\mathbf{E}_z + \frac{\delta x}{2}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y}\right)\right].$$

Thus, circulation can be defined by

$$\oint \mathbf{E}dS = \int_{A\to B}\left(\mathbf{E}_x - \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y} - \frac{\delta z}{2}\frac{\partial \mathbf{E}_x}{\partial z}\right)dx \quad (9)$$
$$+\int_{B\to C}\left(\mathbf{E}_y + \frac{\delta x}{2}\frac{\partial \mathbf{E}_y}{\partial x} + \frac{\delta z}{2}\frac{\partial \mathbf{E}_y}{\partial z}\right)dy$$
$$+\int_{B'\to C'}\left(\mathbf{E}_z - \frac{\delta x}{2}\frac{\partial \mathbf{E}_z}{\partial x} - \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y}\right)dz$$
$$-\int_{C\to D}\left(\mathbf{E}_x + \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y} + \frac{\delta z}{2}\frac{\partial \mathbf{E}_x}{\partial z}\right)dx$$
$$-\int_{D\to A}\left(\mathbf{E}_y - \frac{\delta x}{2}\frac{\partial \mathbf{E}_y}{\partial x} - \frac{\delta z}{2}\frac{\partial \mathbf{E}_y}{\partial z}\right)dy$$
$$-\int_{D'\to A'}\left(\mathbf{E}_z + \frac{\delta x}{2}\frac{\partial \mathbf{E}_z}{\partial x} + \frac{\delta y}{2}\frac{\partial \mathbf{E}_x}{\partial y}\right)dz.$$

Solving and simplifying, yields

$$\oint \mathbf{E}dS = -\delta x\frac{\delta y\partial \mathbf{E}_x}{\partial y} - \delta x\frac{\delta z\partial \mathbf{E}_x}{\partial z} \quad (10)$$
$$+\delta y\frac{\delta x\partial \mathbf{E}_y}{\partial x} + \delta y\frac{\delta z\partial \mathbf{E}_y}{\partial z} - \delta z\frac{\delta x\partial \mathbf{E}_z}{\partial x} - \delta z\frac{\delta y\partial \mathbf{E}_z}{\partial y},$$

and reordering, we can write expression (10) as

$$\oint \mathbf{E}dS = \left(\frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y}\right)\delta x\delta y \quad (11)$$
$$+\left(-\frac{\partial \mathbf{E}_x}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial x}\right)\delta x\delta z + \left(\frac{\partial \mathbf{E}_y}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial y}\right)\delta y\delta z.$$

This expression is geometrically analogue to the equation of total area for a rectangular parallelepiped defined in $z = f(x, y)$ by $A_T = 2ab + 2ac + 2bc$. Thus, from expression (11), its geometrical shape is a rectangular hipper-parallelepiped defined in a four-dimensional frame of reference according to the function $w = f(x, y, x)$.

It is possible to derive tetra-vector **E** from the previous expressions. From the vector analysis we have that expression (10) can be changed to another expression purely differential by multiplying the expression (10) by $\mathbf{S}/S^2$, which is the rotational of the electric field **E**. Then, module of *rot***E** in the central point *O* is expressed through the four coordinates $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{E}_z$, $\mathbf{E}_w$. In the most general case, just representing the *w*-coordinate of vector *rot***E**, which we call as $rot_w\mathbf{E}$, its module in the point *O* is given by

$$rot_w\mathbf{E} = \frac{\mathbf{S}}{S}\frac{\oint \mathbf{E}dS}{S} = \lim_{\delta x,\delta y,\delta z\to 0}\frac{\oint \mathbf{E}dS}{\delta x,\delta y,\delta z}, \quad (12)$$

$$rot_w\mathbf{E} = \frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y} - \frac{\partial \mathbf{E}_x}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial x} \quad (13)$$
$$+\frac{\partial \mathbf{E}_y}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial y} = -\mu_0\frac{\partial \mathbf{H}_w}{\partial t},$$

where μ_0 is the constant of magnetic permeability of the free space and $\mathbf{H}_w$ is the magnetic field intensity in the fourth-dimension. Then, we can write an extension of Maxwell–Faraday equation in the differential form to a fourth dimension as

$$rot_w\mathbf{E} = -\frac{\partial \mathbf{B}_w}{\partial t}, \quad (14)$$

where $\mathbf{B}_w$ is the magnetic field in the fourth dimension.
Thus, electric and magnetic fields are related in an invariant form also in a four-dimensional frame of reference, adding to the three-dimensional expression those terms described in function of the *z*-coordinate, which are

$$\frac{\partial \mathbf{E}_y}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial y} - \frac{\partial \mathbf{E}_x}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial x}. \quad (15)$$

Then, without such additional terms, expression (14) is reduced to the three-dimensional form as defined in the classical Maxwell-Faraday expression (2), given by

$$rot_z\mathbf{E} = \frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y} = -\mu_0\frac{\partial \mathbf{H}_z}{\partial t} = -\frac{\partial \mathbf{B}_z}{\partial t}. \quad (16)$$

III. CONCLUSIONS

Maxwell-Faraday equation is one of the fundamental expressions of the classical electrodynamics theory, developed in a three-dimensional frame of reference. Nevertheless, considering other possible higher dimensions (for instance, the fourth dimension) allows one to consider the possibility to explore the theoretical extension of electrodynamics to such higher dimensions, not only from a tensor calculation or another complex mathematical method, but also applying the classical method as the directional derivatives. Thus, extending the Maxwell-Faraday equation to a four-dimensional frame of reference by the method of directional derivatives, allows to suppose (at less theoretically) that electromagnetic field can be unified also in a four-dimensional frame of reference.

Regarding to education, it is revisited the classical Faraday's law showing the method of directional derivatives to derive Maxwell-Faraday equation in its differential form, but now adding an extra dimensional parameter in order to extend such deduction to a four-dimensional frame of reference, which shows the applicability of the classical methods also in the additional dimensions. In the same way, it could be possible to apply a similar method in order to extend to four-dimensions some of the remaining Maxwell's equations (for instance, Ampère's law with Maxwell's extension).

ACKNOWLEDGEMENTS

The author would like to thank Professor Sergio S. Cornejo for the review and comments for this paper.

REFERENCES

[1] Ulaby, F. T., *Fundamentals of Applied Electromagnetics*, 5th Ed. (Prentice Hall, New Jersey, USA, 2007), pp. 257-285.

[2] Jackson, J. D., *Classical Electrodynamics*, 2nd Ed. (Wiley, New York, 1975), pp. 580-648.

[3] Born, M. and Wolf, E., *Principles of Optics*, 6th Ed. (Cambridge University Press, Cambridge, 1980), pp. 4-8.

[4] Maxwell, J. C., *A Treatise on Electricity and Magnetism*, Vol. II, 3rd Ed. (Oxford University Press, UK, 1904), pp. 178-179 and 189.

[5] Hinton, C. H., *The Four Dimensions*, 3rd Ed. (George Allen & Co. LTD, London, 1912) pp. 15-75.

[6] Aflalo, T. N. G., *Four-Dimensional Spatial Reasoning in Humans*, Journal of Experimental Psychology: Human Perception and Performance **34**, 1066-1077 (2008).

[7] Lakhtakia, A., *Essays on the Formal Aspects of Electromagnetic Theory*, (World Scientific Publishing Co. Pte. Ltd., Singapore, 1993), pp. 142–147.

[8] Marsden, J. E. and Hughes, T. J. R., *Mathematical Foundations of Elasticity*, (Prentice Hall, New Jersey, USA, 1983), pp. 36-40.

[9] Hildebrand, F. B., *Advanced Calculus for Applications*, 2nd Ed. (Prentice Hall, New Jersey, USA, 1976).

[10] Leithold, L., *The Calculus with Analytical Geometry*, 2nd Ed. (Harper & Row, Publishers, Inc., New York, 1972), pp. 886-924.