Jerk, curvature and torsion in motion of charged particle under electric and magnetic fields

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Abstract

The existence of the jerk vector and the resulting curvature and torsion are investigated in the motion of a charged particle under the influence uniform of electric and magnetic fields. It is shown that a uniform electric field, by itself, does not produce jerk and hence cannot create torsion. It produces curvature if the initial velocity of the charged particle has a component perpendicular to the electric field. A uniform magnetic field, on the other hand, produces jerk motion, and curvature and torsion of constant magnitudes. The perpendicular and parallel components of the initial velocity of the particle are responsible for the curvature and torsion, respectively. When electric and magnetic fields parallel with one another are present, they both contribute to curvature and torsion. The path is a helix which is continuously being stretched out. As a result, both curvature and torsion approach zero as time progresses.

Keywords: Jerk vector, Curvature, Torsion.

Resumen

La existencia del vector Jerk y la curvatura y torsión resultantes son investigadas en el movimiento de una partícula cargada bajo la influencia de los campos eléctricos y magnéticos. Se ha demostrado que un campo eléctrico uniforme, por sí solo, no produce Jerk y por lo tanto no puede crear torsión. Esto produce una curvatura si la velocidad inicial de la partícula cargada tiene una componente perpendicular al campo eléctrico. Un campo eléctrico uniforme, por el contrario, produce un movimiento Jerk, y la curvatura y torsión de las magnitudes constantes. Los componentes perpendicular y paralela de la velocidad inicial de la partícula son responsables de la curvatura y la torsión respectivamente. Cuando los campos eléctricos y magnéticos paralelos entre sí están presentes, estos contribuyen a la curvatura y la torsión. El camino es una hélice que está siendo continuamente extendida. Como resultado de ello, tanto la curvatura y la torsión tienden a cero, como pasa el tiempo.

Palabras clave: Vector Jerk, Curvatura, Torsión.

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I. INTRODUCTION

One of the relatively neglected topics in physics is the *jerk vector*. Formerly known as the *second acceleration*, it is the derivative of the acceleration vector, or the third derivative of the position vector, with respect to time. Nonetheless, the jerk vector has been studied in simple harmonic motion [1], uniform circular motion [2], Keplerian motion [1, 3], and projectile motion [4]. In this paper, we investigate the existence of the jerk vector in the motion of a charged particle under the action of uniform electric and magnetic fields.

Two other concepts related to a curve and seldom mentioned in physics are the *curvature* and *torsion*. The curvature is the arc-rate of turning of the tangent vector in a plane, whereas the torsion, formerly called the *second curvature*, is the arc-rate of turning of the tangent out of the plane [5, 6]. These concepts of differential geometry are well-suited to dynamical problems when the curve under consideration is the trajectory of a particle. If the first three derivatives of the position vector in time, viz., the velocity, acceleration and the jerk vectors are $\vec{v}$, $\vec{a}$ and $\vec{j}$, respectively, then the curvature κ and torsion τ are given by [7]:

$$\kappa = \frac{\left|\vec{v} \times \vec{a}\right|}{\left|\vec{v}\right|^3}, \tag{1}$$

and

$$\tau = \frac{\left|\vec{v} \circ \left(\vec{a} \times \vec{j}\right)\right|}{\left|\vec{v} \times \vec{a}\right|^2}. \tag{2}$$

Using the above formulas, one can conveniently calculate the jerk, curvature and torsion for the charged particle motion under the influence of electric and magnetic fields. The equation of motion of a charged particle of mass m and electric charge q under the electric field $\vec{E}$ and magnetic field $\vec{B}$ is given by the Lorentz equation. In Gaussian system of units, we have:

$$m\frac{d\vec{v}}{dt} = q\vec{E} + q\vec{v}\times\vec{B}\,. \qquad (3)$$

II. MOTION OF CHARGED PARTICLE IN UNIFORM ELECTRIC FIELD

We first investigate the motion of a charged particle in a uniform electric field. In this and the following ezamples, we consider a positive charge ($q > 0$). Choose the electric field in the positive x-direction: $\vec{E} = E\hat{x}$. Without loss of generality, choose the initial velocity of the particle as $\vec{v}_0 = v_{II}\hat{x} + v_{\perp}\hat{y}$. Further, let the initial position of the particle be at the origin. The equations of motion are the following:

$$\frac{dv_x}{dt} = \Lambda\,, \qquad (4)$$

$$\frac{dv_y}{dt} = 0\,, \qquad (5)$$

and

$$\frac{dv_z}{dt} = 0\,. \qquad (6)$$

where $\Lambda = qE/m$. Integrating Eqs. (4-6) twice with respect to time, we get:

$$v_x = \Lambda t + C_1\,, \qquad (7)$$

$$x = \frac{1}{2}\Lambda t^2 + C_1 t + C_2\,, \qquad (8)$$

$$v_y = C_3\,, \qquad (9)$$

$$y = C_3 t + C_4\,, \qquad (10)$$

$$v_z = C_5\,, \qquad (11)$$

and

$$z = C_5 t + C_6\,. \qquad (12)$$

where C_1, C_2, etc., are the constants of integration. The initial conditions give: $C_1 = v_{II}$; $C_2 = 0$; $C_3 = v_{\perp}$; $C_4 = 0$; $C_5 = C_6 = 0$. Thus the motion is entirely in the x - y plane with

$$x = v_{II} t + \frac{1}{2}\Lambda t^2\,, \qquad (13)$$

and

$$y = v_{\perp} t\,. \qquad (14)$$

Eliminating t between Eqs. (13) and (14), we get

$$x = \frac{v_{II}}{v_{\perp}} y + \frac{\Lambda}{2{v_{\perp}}^2} y^2\,. \qquad (15)$$

Thus the path of the charged particle is a parabola (Fig. 1).

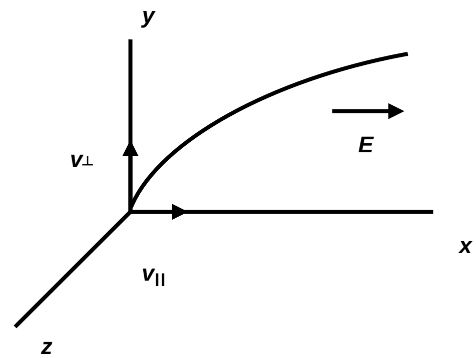

FIGURE 1. Charged particle in a uniform electric field.

The acceleration and jerk vectors are obtained by successive differentiation of the velocity vector, giving:

$$\vec{v} = (v_{II} + \Lambda t)\hat{x} + v_{\perp}\hat{y}\,, \qquad (16)$$

$$\vec{a} = \Lambda\hat{x}\,, \qquad (17)$$

and

$$\vec{j} = \vec{0}\,. \qquad (18)$$

The curvature and torsion can readily be calculated using Eqs. (1) and (2), giving:

$$\kappa = \frac{\Lambda v_{\perp}}{\left[(v_{II} + \Lambda t)^2 + {v_{\perp}}^2\right]^{3/2}}\,, \qquad (19)$$

and

$$\tau = 0\,. \qquad (20)$$

Hence, *an electric field can create curvature in the motion of a charged particle but not torsion*. From Eq. (19), it follows that as $t \to \infty$, $\kappa \to 0$, *i.e.*, the curvature diminishes in time. Also, if $v_{\perp} = 0$, then $\kappa = 0$. Thus, *it is the perpendicular component of the initial velocity which is responsible for producing the curvature.*

III. MOTION OF CHARGED PARTICLE IN UNIFORM MAGNETIC FIELD

We next investigate the motion of a charged particle in a uniform magnetic field. Choose the magnetic field in the positive x-direction: $\vec{B} = B\hat{x}$. Without loss of generality, choose the initial velocity of the particle as

$\vec{v}_0 = v_{II}\hat{x} + v_{\perp}\hat{y}$. Further, let the initial position vector of the particle be $\vec{r} = z_0\hat{z}$. The equations of motion are the following:

$$\frac{dv_x}{dt} = 0, \tag{21}$$

$$\frac{dv_y}{dt} = \Omega v_z, \tag{22}$$

and

$$\frac{dv_z}{dt} = -\Omega v_y. \tag{23}$$

where $\Omega = qB/m$ is the gyrofrequency. The solutions to Eq. (21) subject to the initial conditions are:

$$v_x = v_{II}, \tag{24}$$

and

$$x = v_{II}t. \tag{25}$$

Eqs. (23) and (24) are coupled equations. Differentiating each and substituting from the other, one gets:

$$\frac{d^2v_y}{dt^2} + \Omega^2 v_y = 0, \tag{26}$$

and

$$\frac{d^2v_z}{dt^2} + \Omega^2 v_z = 0. \tag{27}$$

Let the solution to Eq. (26) be

$$v_y = C_1 \sin\Omega t + C_2 \cos\Omega t, \tag{28}$$

By differentiation and substitution from Eq. (23):

$$v_z = C_1 \cos\Omega t - C_2 \sin\Omega t. \tag{29}$$

The initial conditions furnish: $C_1 = 0$; and $C_2 = v_{\perp}$. Thus

$$v_y = v_{\perp} \cos\Omega t, \tag{30}$$

and

$$v_z = -v_{\perp} \sin\Omega t. \tag{31}$$

Integrating and applying initial conditions, we get

$$y = \frac{v_{\perp}}{\Omega}\sin\Omega t, \tag{32}$$

and

$$z = \frac{v_{\perp}}{\Omega}\cos\Omega t. \tag{33}$$

Combining Eqs. (24, 25, 30, 31, 33) and (34):

$$\vec{r} = v_{II}t\hat{x} + \frac{v_{\perp}}{\Omega}\sin\Omega t\hat{y} + \frac{v_{\perp}}{\Omega}\cos\Omega t\hat{z}, \tag{34}$$

and

$$\vec{v} = v_{II}\hat{x} + v_{\perp}\cos\Omega t\hat{y} - v_{\perp}\sin\Omega t\hat{z}. \tag{35}$$

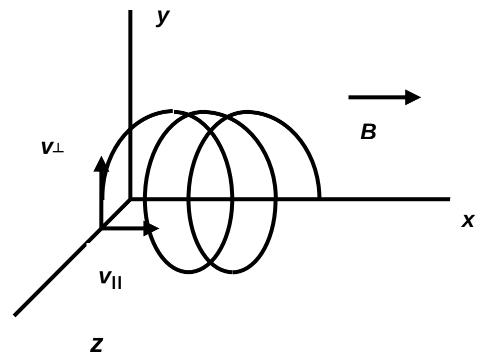

FIGURE 2. Charged particle in a uniform magnetic field.

Eq. (34) is the parametric equation of a helix about the *x*-axis having radius $v_{\perp}/\Omega$ and pitch $v_{II}t$ (Fig. 2).

The acceleration and jerk vectors are obtained by successive differentiation of the velocity vector, giving:

$$\vec{a} = -v_{\perp}\Omega\sin\Omega t\hat{y} - v_{\perp}\Omega\cos\Omega t\hat{z}, \tag{36}$$

and

$$\vec{j} = -v_{\perp}\Omega^2\cos\Omega t\hat{y} + v_{\perp}\Omega^2\sin\Omega t\hat{z}. \tag{37}$$

The curvature and torsion can readily be calculated using Eqs. (1) and (2). Upon carrying out the calculations, one obtains:

$$|\vec{v}\times\vec{a}| = \Omega v_{\perp}\sqrt{v_{II}^2 + v_{\perp}^2}, \tag{38}$$

$$|\vec{v}|^3 = \left(v_{II}^2 + v_{\perp}^2\right)^{3/2}, \tag{39}$$

and

$$\left|\vec{v}\circ\left(\vec{a}\times\vec{j}\right)\right| = \Omega^3 v_{II} v_{\perp}^2, \tag{40}$$

whence

$$\kappa = \frac{\Omega v_{\perp}}{v_{II}^2 + v_{\perp}^2} = \frac{\Omega v_{\perp}}{v_0^2}, \tag{41}$$

and

$$\tau = \frac{\Omega v_{II}}{v_{II}^2 + v_{\perp}^2} = \frac{\Omega v_{II}}{v_0^2}. \tag{42}$$

Eqs. (41) and (42) indicate that *a uniform magnetic field can produce both curvature and torsion in the motion of a charged particle*. The equations further show that if $v_{\perp} = 0$, then $\kappa = 0$; and if $v_{II} = 0$, then $\tau = 0$. *Thus, the* *perpendicular and parallel components of the initial velocity of the particle are responsible for the curvature and torsion, respectively*. The curvature bears a constant ration to the torsion equal to $v_\perp / v_{II}$, which is the tangent of the angle the initial velocity makes with the magnetic field.

IV. MOTION OF CHARGED PARTICLE IN PARALLEL ELECTRIC AND MAGNETIC FIELDS

In our next example, we consider the motion of a charged particle under the combined actions of parallel electric and magnetic fields. Let $\vec{E} = E\hat{x}$ and $\vec{B} = B\hat{x}$. As before, let the initial velocity of the particle be $\vec{v}_0 = v_{II}\hat{x} + v_\perp \hat{y}$. Further, let the initial position vector of the particle be $\vec{r} = z_0 \hat{z}$. The equations of motion are the following:

$$\frac{dv_x}{dt} = \Lambda , \tag{43}$$

$$\frac{dv_y}{dt} = \Omega v_z , \tag{44}$$

and

$$\frac{dv_z}{dt} = -\Omega v_y . \tag{45}$$

Eqs. (43–45) can be integrated following our earlier procedure to yield:

$$\vec{r} = \left(v_{II} t + \frac{1}{2}\Lambda t^2 \right)\hat{x} + \frac{v_\perp}{\Omega}\sin \Omega t\hat{y} + \frac{v_\perp}{\Omega}\cos \Omega t\hat{z} , \tag{46}$$

$$\vec{v} = \left(v_{II} + \Lambda t \right)\hat{x} + v_\perp \cos \Omega t\hat{y} - v_\perp \sin \Omega t\hat{z} , \tag{47}$$

$$\vec{a} = \Lambda\hat{x} - v_\perp \Omega \sin \Omega t\hat{y} - v_\perp \Omega \cos \Omega t\hat{z} , \tag{48}$$

and

$$\vec{j} = -v_\perp \Omega^2 \cos \Omega t\hat{y} + v_\perp \Omega^2 \sin \Omega t\hat{z} . \tag{49}$$

Eq. (46) describes a helix about the *x*-axis, whose pitch is continuously being stretched out in time.

The curvature and torsion can be calculated using Eqs. (1) and (2) as before. Upon carrying out the calculations, one obtains:

$$\left| \vec{v} \times \vec{a} \right| = v_\perp \sqrt{\Omega^2 v_\perp{}^2 + \Omega^2 \left(v_{II} + \Lambda t \right)^2 + \Lambda^2} , \tag{50}$$

$$\left| \vec{v} \right|^3 = \left[v_\perp{}^2 + \left(v_{II} + \Lambda t \right)^2 \right]^{3/2} , \tag{51}$$

and

$$\left| \vec{v} \circ \left(\vec{a} \times \vec{j} \right) \right| = \Omega^3 v_\perp{}^2 \left(v_{II} + \Lambda t \right), \tag{52}$$

whence

$$\kappa = \frac{v_\perp \sqrt{\Omega^2 v_\perp{}^2 + \Omega^2 \left(v_{II} + \Lambda t \right)^2 + \Lambda^2}}{\left[v_\perp{}^2 + \left(v_{II} + \Lambda t \right)^2 \right]^{3/2}} , \tag{53}$$

and

$$\tau = \frac{\Omega^3 \left(v_{II} + \Lambda t \right)}{\Omega^2 v_\perp{}^2 + \Omega^2 \left(v_{II} + \Lambda t \right)^2 + \Lambda^2} . \tag{54}$$

Eqs. (53) and (54) give the curvature and torsion in the trajectory of a charged particle under the combined action of parallel electric and magnetic fields. *It is apparent that both the electric field E* (through Λ) *and the magnetic field B* (through Ω) *are contributory to the curvature and torsion. Thus, even though the electric field alone cannot produce torsion, it can modify the torsion produced by a magnetic field*. Eq. (53) shows that if $v_\perp = 0$, then $\kappa = 0$. *Thus, the perpendicular component of the initial velocity of the particle is responsible for producing the curvature*.

One can verify that: (1) If $B = 0$, *i.e.*, $\Omega = 0$, then Eqs. (53) and (54) reduce to Eqs. (19) and (20), respectively; and (2) If $E = 0$, *i.e.*, $\Lambda = 0$, then Eqs. (53) and (54) reduce to Eqs. (41) and (42), respectively. In other words, the first two examples follow as special cases of the more general example. It is suggested that as a follow-up study, other general cases of charged particle motion (*e.g.*, under the action of mutually perpendicular electric and magnetic fields) be carried out.

V. CONCLUSIONS

The topics of jerk, curvature and torsion are not part of the normal curriculum and are seldom discussed in the literature. This paper demonstrates that they are useful concepts which can be applied and illustrated in common examples in physics.

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