# A concise self-closed way to the acceleration formula for the moving frame

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**Abstract**
The motion analysis of a particle in a moving frame is very important in courses of both physics and engineering mechanics. Nonetheless, the narrations in most textbooks are not completely satisfactory. This report suggests a concise approach to derive the acceleration formula for a moving frame. The suggested approach, starting from the unit moving vector, is self-closed. The deduction makes uses of the vector, which leads to a clear image of the Coriolis acceleration.

**Keywords:** Kinematics, non-inertial frame, Coriolis acceleration, vector operation, composite motion.

**Resumen**
El análisis del movimiento de una partícula en un sistema en movimiento es muy importante en los cursos de física y de ingeniería mecánica. Sin embargo, las narraciones en la mayoría de los libros de texto no son completamente satisfactorias. Este informe propone un enfoque conciso para obtener la fórmula de la aceleración de un sistema en movimiento. El enfoque propuesto, a partir del vector unitario de movimiento, es auto-contenido. La deducción hace uso del vector, el cual conduce a una imagen clara de la aceleración de Coriolis

**Palabras clave:** Cinemática, no inercial, la aceleración de Coriolis, la operación de vectores, movimiento compuesto.

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## I. INTRODUCTION

The kinematics of the non-inertial frame is an essential component in textbooks of both physics (*e.g.* [1, 2, 3]) and engineering mechanics (*e.g.* [4, 5, 6]). Most college physics textbooks illustrate the relevant knowledge by empirical examples without systematic deduction (*e.g.* [6, 7]), especially the concept of the Coriolis acceleration. Textbooks in engineering mechanics strive to go into depth, but the ways of knowledge delivery and argument are not always completely satisfactory. Some start from the derivative of a unit vector (*e.g.* [5]) while others start from the derivative of a length-constant vector (*e.g.* [8]). There are other common imperfections in the regularly-used textbooks. For example: 1) Only the planar case is expounded for the sake of brevity (*e.g.* [9]). 2) The deduction is complicated with projecting the vector to the axes of moving, although the vector operation, rather than the scalar operation, is highlighted in most textbooks. This projection leads to a tedious procedure. And 3) the physical meaning of the Coriolis acceleration is vague in the projection-based deduction (*e.g.* [4, 5, 8, 10]).

This report presents a concise and self-closed approach to derive the acceleration formula for a moving frame, with the unit moving vector as the start entry. The deduction makes uses of vectorial geometry, which leads to a clear image of the Coriolis acceleration.

## II. DEVIRATIVES OF UNIT VECTORS

A moving frame $O'x'y'z'$ moves in a fixed frame $Oxyz$, as sketched in Fig. 1. $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the three unit vectors of the $x$-, $y$- and $z$- axes of the fixed frame, respectively. $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ are the corresponding unit vectors of the moving frame. We first determine the derivatives of $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$. Because of moving, their derivatives are not zero, which contrasts against the always-zero derivatives of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

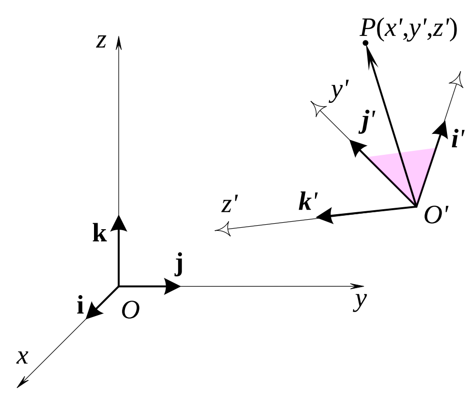


**FIGURE 1.** Moving Frame.

First we know the length of $\boldsymbol{i}'$ is fixed to be one, thus

$$\boldsymbol{i}'\cdot\boldsymbol{i}'=1. \quad (1)$$

Differentiating both sides of Eq. (1) leads to

$$\frac{\mathrm{d}\boldsymbol{i}'}{\mathrm{d}t}\cdot\boldsymbol{i}'+\boldsymbol{i}'\cdot\frac{\mathrm{d}\boldsymbol{i}'}{\mathrm{d}t}=0.$$

Because of the exchangeability of the vector product, the above equation can be rewritten as

$$\frac{\mathrm{d}\boldsymbol{i}'}{\mathrm{d}t}\cdot\boldsymbol{i}'=0. \quad (2)$$

Eq. (2) indicates that the derivative of $\boldsymbol{i}'$ is perpendicular to itself. From the physical view, $\mathrm{d}\boldsymbol{i}'/\mathrm{d}t$ is the relative velocity of the endpoint of $\boldsymbol{i}'$ relative to the moving $O'$. The vector $\boldsymbol{i}'$ has a fixed length of one, so the relative moving path of its endpoint with respect to the moving frame $O'x'y'z'$ falls on a unit sphere surface, as illustrated in Fig. 2. The velocity direction is tangent to the sphere surface, accordingly, perpendicular to the radius $\boldsymbol{i}'$ always.

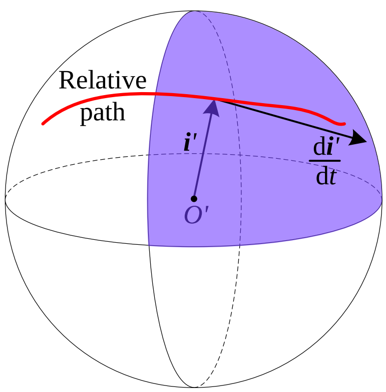


**FIGURE 2.** Relative moving path on a sphere surface.

Eq. (2) implies that $\mathrm{d}\boldsymbol{i}'/\mathrm{d}t$ can be expressed as some vector cross-multiplying with $\boldsymbol{i}'$. We denote this unknown vector as $\boldsymbol{\omega}_{i'}$, that is

$$\frac{\mathrm{d}\boldsymbol{i}'}{\mathrm{d}t}=\boldsymbol{\omega}_{i'}\times\boldsymbol{i}'. \quad (3)$$

Clearly, Eq. (3) solely cannot determine the $\boldsymbol{i}'$ component of $\boldsymbol{\omega}_{i'}$ uniquely, since varying the $\boldsymbol{i}'$ component does not violate Eq. (3).

Similarly, we have the following equations for $\boldsymbol{j}'$ and $\boldsymbol{k}'$ from $\boldsymbol{j}'\cdot\boldsymbol{j}'=1$ and $\boldsymbol{k}'\cdot\boldsymbol{k}'=1$,

$$\frac{\mathrm{d}\boldsymbol{j}'}{\mathrm{d}t}=\boldsymbol{\omega}_{j'}\times\boldsymbol{j}', \quad (4)$$

$$\frac{\mathrm{d}\boldsymbol{k}'}{\mathrm{d}t}=\boldsymbol{\omega}_{k'}\times\boldsymbol{k}'. \quad (5)$$

The subscripts of $\boldsymbol{\omega}_{i'}$, $\boldsymbol{\omega}_{j'}$ and $\boldsymbol{\omega}_{k'}$ in Eq. (3)~Eq. (5) accentuate their dependence on the specific original vectors. They contain nine components in form altogether (three for each 3D-vector). However these components are not free. There are three constraints among them. Using these constraints, we can eventually choose

$$\boldsymbol{\omega}_{i'}=\boldsymbol{\omega}_{j'}=\boldsymbol{\omega}_{k'}=\boldsymbol{\omega}, \quad (6)$$

where the rightmost $\boldsymbol{\omega}$ is independent of the original vector.

The first constraint is that the moving frame of $O'x'y'z'$ is a rigid body, so $\boldsymbol{i}'$ and $\boldsymbol{j}'$ are always perpendicular to each other. That is

$$\boldsymbol{i}'\cdot\boldsymbol{j}'=0. \quad (7)$$

Differentiating both sides of Eq. (7) leads to

$$\frac{\mathrm{d}\boldsymbol{i}'}{\mathrm{d}t}\cdot\boldsymbol{j}'+\boldsymbol{i}'\cdot\frac{\mathrm{d}\boldsymbol{j}'}{\mathrm{d}t}=0. \quad (8)$$

Substituting Eq. (3) and Eq. (4) into Eq. (8) leads to

$$(\boldsymbol{\omega}_{i'}\times\boldsymbol{i}')\cdot\boldsymbol{j}'+\boldsymbol{i}'\cdot(\boldsymbol{\omega}_{j'}\times\boldsymbol{j}')=0. \quad (9)$$

According to the law for vector operations, Eq. (9) can be reformulated as

$$\boldsymbol{\omega}_{i'}\cdot(\boldsymbol{i}'\times\boldsymbol{j}')+\boldsymbol{\omega}_{j'}\cdot(\boldsymbol{j}'\times\boldsymbol{i}')=0. \quad (10)$$

Substituting $\boldsymbol{i}'\times\boldsymbol{j}'=-\boldsymbol{j}'\times\boldsymbol{i}'=\boldsymbol{k}'$ into Eq. (10) leads to

$$\boldsymbol{\omega}_{i'}\cdot\boldsymbol{k}'=\boldsymbol{\omega}_{j'}\cdot\boldsymbol{k}'. \quad (11)$$

Eq. (11) shows that the $\boldsymbol{k}'$ components of $\boldsymbol{\omega}_{i'}$ and $\boldsymbol{\omega}_{j'}$ are the same. The other two constraints can be followed similarly from $\boldsymbol{j}'\cdot\boldsymbol{k}'=0$ and $\boldsymbol{k}'\cdot\boldsymbol{i}'=0$,

$$\boldsymbol{\omega}_{i'}\cdot\boldsymbol{j}'=\boldsymbol{\omega}_{k'}\cdot\boldsymbol{j}', \quad (12)$$

$$\boldsymbol{\omega}_{j'}\cdot\boldsymbol{i}'=\boldsymbol{\omega}_{k'}\cdot\boldsymbol{i}'. \quad (13)$$

Eq. (11)~Eq. (13) can be summarized in Table I (Same symbols corresponding to equal-value constraints). The table shows that the two non-diagonal components in same column are identical. The diagonal components do not involve any constraint. They can be arbitrarily chosen without violating Eq. (3)-Eq. (5). Thus all the three diagonal components can be made to equal the non-diagonal individuals. Then all the three rows in Table I are the same, and the three components in a single row makes up a unique vector $\boldsymbol{\omega}$ as in Eq. (6).

**TABLE I.** The constraints among nine components.

| | $\boldsymbol{i}'$ | $\boldsymbol{j}'$ | $\boldsymbol{k}'$ |
|---|---|---|---|
| $\boldsymbol{\omega}_{i'}$ | X | ▼ | Δ |
| $\boldsymbol{\omega}_{j'}$ | ► | Y | Δ |
| $\boldsymbol{\omega}_{k'}$ | ► | ▼ | Z |

## III. DEVIRATIVES OF LENGTH-CONSTANT VECTORS

For any point $P$ fixed to the moving frame (Fig. 1), we have

$$\frac{\mathrm{d}\overline{O'P}}{\mathrm{d}t}=\boldsymbol{\omega}\times\overline{O'P}\,. \tag{14}$$

**Proof**

Substituting $\overline{O'P}=x'\boldsymbol{i}'+y'\boldsymbol{j}'+z'\boldsymbol{k}'$ into the left hand side of Eq. (14), we have

$$\begin{aligned}\frac{\mathrm{d}\overline{O'P}}{\mathrm{d}t}&=\frac{\mathrm{d}(x'\boldsymbol{i}'+y'\boldsymbol{j}'+z'\boldsymbol{k}')}{\mathrm{d}t}\\&=x'\frac{\mathrm{d}\boldsymbol{i}'}{\mathrm{d}t}+\frac{\mathrm{d}x'}{\mathrm{d}t}\boldsymbol{i}'+y'\frac{\mathrm{d}\boldsymbol{j}'}{\mathrm{d}t}+\frac{\mathrm{d}y'}{\mathrm{d}t}\boldsymbol{j}'+z'\frac{\mathrm{d}\boldsymbol{k}'}{\mathrm{d}t}+\frac{\mathrm{d}z'}{\mathrm{d}t}\boldsymbol{k}'\end{aligned}$$

Since the $P$ fixed to the moving frame has invariant ($x'$, $y'$, $z'$), we have $\frac{\mathrm{d}x'}{\mathrm{d}t}=\frac{\mathrm{d}y'}{\mathrm{d}t}=\frac{\mathrm{d}z'}{\mathrm{d}t}=0$. Accordingly,

$$\frac{\mathrm{d}\overline{O'P}}{\mathrm{d}t}=x'\frac{\mathrm{d}\boldsymbol{i}'}{\mathrm{d}t}+y'\frac{\mathrm{d}\boldsymbol{j}'}{\mathrm{d}t}+z'\frac{\mathrm{d}\boldsymbol{k}'}{\mathrm{d}t}\,. \tag{15}$$

Substituting Eq. (3)-Eq. (6) into Eq. (15) leads to

$$\begin{aligned}\frac{\mathrm{d}\overline{O'P}}{\mathrm{d}t}&=x'\boldsymbol{\omega}\times\boldsymbol{i}'+y'\boldsymbol{\omega}\times\boldsymbol{j}'+z'\boldsymbol{\omega}\times\boldsymbol{k}'\\&=\boldsymbol{\omega}\times(x'\boldsymbol{i}'+y'\boldsymbol{j}'+z'\boldsymbol{k}')\end{aligned}.$$

The right hand side (RHS) of the second equal sign is exactly that of the Eq. (14).

If a vector $\overline{PQ}$ is fixed to the moving frame, then we have

$$\begin{aligned}\frac{\mathrm{d}\overline{PQ}}{\mathrm{d}t}&=\frac{\mathrm{d}(\overline{O'Q}-\overline{O'P})}{\mathrm{d}t}=\frac{\mathrm{d}\overline{O'Q}}{\mathrm{d}t}-\frac{\mathrm{d}\overline{O'P}}{\mathrm{d}t}\,,\\&=\boldsymbol{\omega}\times\overline{O'Q}-\boldsymbol{\omega}\times\overline{O'P}\,.\end{aligned}$$

That is

$$\frac{\mathrm{d}\overline{PQ}}{\mathrm{d}t}=\boldsymbol{\omega}\times\overline{PQ}\,. \tag{16}$$

Eq. (14) and Eq. (16) shows that $\boldsymbol{\omega}$ reflects the motion quantity of the entire moving frame. When analysis deteriorates to the planar case, $\boldsymbol{\omega}$ corresponds to the scalar quantity of angular velocity $\omega$. Thus $\boldsymbol{\omega}$ is termed the angular velocity vector.

## IV. ACCELERATION ANALYSIS

Fig. 3 illustrates the relative motion of a particle. The relative path is fixed to the moving frame $O'x'y'z'$. The particle moves along the relative path in the moving frame. The moving frame, carrying the relative path and the particle, moves in the fixed frame $Oxyz$.

At instant $t$, the particle is on the point $A$ as illustrated in Fig. 3(a). It moves to the point $B'$ after $\Delta t$, as shown in Fig. 3(b).

The absolute acceleration is

$$\boldsymbol{a}_{\mathrm{a}}=\lim_{\Delta t\to 0}\frac{\boldsymbol{v}_{B'\mathrm{a}}-\boldsymbol{v}_{A\mathrm{a}}}{\Delta t}\,, \tag{17}$$

where $\boldsymbol{v}_{A\mathrm{a}}$, $\boldsymbol{v}_{B\mathrm{a}}$ are the absolute velocities at instant $t$ and $t+\Delta t$, respectively.

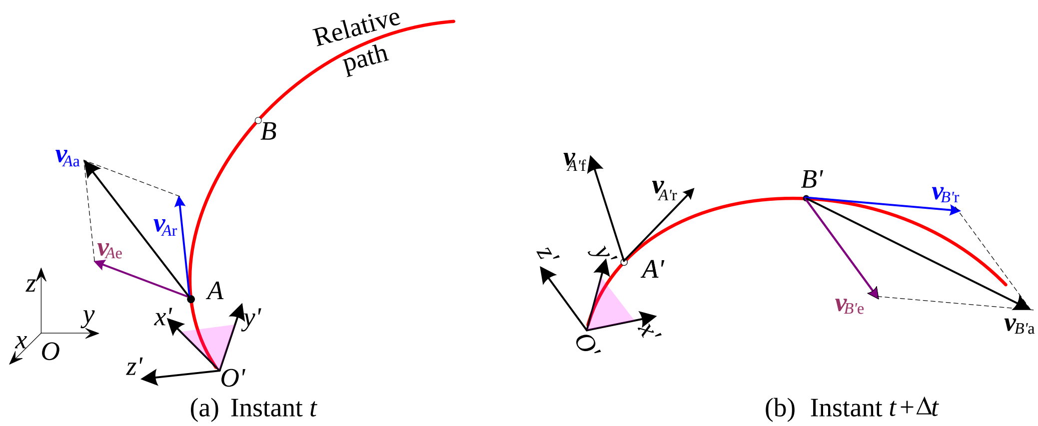


**FIGURE 3.** Relative moving path on a sphere surface.

According to relative velocity analysis (Fig. 3), they can be expressed as

$$\boldsymbol{v}_{Aa} = \boldsymbol{v}_{Ar} + \boldsymbol{v}_{Ae}, \boldsymbol{v}_{B'a} = \boldsymbol{v}_{B'r} + \boldsymbol{v}_{B'e}, \tag{18}$$

where $\boldsymbol{v}_{Ar}, \boldsymbol{v}_{B'r}$ are the relative velocities at instant $t$ and $t+\Delta t$, respectively. $\boldsymbol{v}_{Ae}, \boldsymbol{v}_{B'e}$ in Eq. (18) are the entrained velocities at instant $t$ and $t+\Delta t$, respectively. The so called "entrained point" is the point fixed in the moving frame but overlapping with the particle temporally. The entrained velocities are the velocities of those points.

Substituting Eq. (18) into Eq. (17) leads to

$$\boldsymbol{a}_a = \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'r} + \boldsymbol{v}_{B'e} - (\boldsymbol{v}_{Ar} + \boldsymbol{v}_{Ae})}{\Delta t}.$$

It can be recombined into

$$\boldsymbol{a}_a = \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'r} - \boldsymbol{v}_{Ar}}{\Delta t} + \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'e} - \boldsymbol{v}_{Ae}}{\Delta t}. \tag{19}$$

It should be pointed out that the first term on the RHS of Eq. (19) is NOT the relative acceleration. The reason is that the relative velocity sensed in the moving frame must be tangential to the moving path. The relative velocity (at instant $t$) we sense (at instant $t+\Delta t$) is $\boldsymbol{v}_{A'r}$ (it must be tangential to the moving path) in Fig. 3(b), rather than $\boldsymbol{v}_{Ar}$ in Fig. 3(a). Thus

$$\lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'r} - \boldsymbol{v}_{Ar}}{\Delta t} = \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'r} - \boldsymbol{v}_{A'r} + \boldsymbol{v}_{A'r} - \boldsymbol{v}_{Ar}}{\Delta t}.$$

That is

$$\lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'r} - \boldsymbol{v}_{Ar}}{\Delta t} = \boldsymbol{a}_r + \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{A'r} - \boldsymbol{v}_{Ar}}{\Delta t}. \tag{20}$$

Assuming you are in the moving frame, your sense of the $\boldsymbol{v}_{Ar}$ occurring at $t$ does not vary after the time $t$. But viewed from the fixed frame, $\boldsymbol{v}_{Ar}$ at $t$ is a length-fixed vector moving as the moving frame. The second term on the RHS of Eq. (20 is just the derivative of such a vector. According to Eq. (16), we have

$$\lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{A'r} - \boldsymbol{v}_{Ar}}{\Delta t} = \boldsymbol{\omega} \times \boldsymbol{v}_{Ar}. \tag{21}$$

Now we turn to the second term on the RHS of Eq. (19). This is NOT the entrained acceleration either. The entrained point at instant $t$ moves to $A'$ at instant $t+\Delta t$ *. The entrained acceleration at instant $t$ should be

$$\boldsymbol{a}_e = \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{A'f} - \boldsymbol{v}_{Ae}}{\Delta t}, \tag{22}$$

where $\boldsymbol{v}_{A'f}$ is the velocity of the point $A'$ fixed to the moving frame.

With $\boldsymbol{v}_{A'f}$, the second term on the RHS of Eq. (19) can be rewritten as

$$\begin{aligned}\lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'e} - \boldsymbol{v}_{Ae}}{\Delta t} &= \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'e} - \boldsymbol{v}_{A'f} + \boldsymbol{v}_{A'f} - \boldsymbol{v}_{Ae}}{\Delta t} \\ &= \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'e} - \boldsymbol{v}_{A'f}}{\Delta t} + \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{A'f} - \boldsymbol{v}_{Ae}}{\Delta t}\end{aligned}.$$

That is

$$\lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'e} - \boldsymbol{v}_{Ae}}{\Delta t} = \lim_{\Delta t\to 0} \frac{\boldsymbol{v}_{B'e} - \boldsymbol{v}_{A'f}}{\Delta t} + \boldsymbol{a}_e. \tag{23}$$

* At instant $t+\Delta t$, $A'$ or $A$ is no longer the entrained point. The new entrained point is $B$.

$\boldsymbol{v}_{B'\mathrm{e}}$, $\boldsymbol{v}_{A'\mathrm{f}}$ in Eq. (23) are the velocities of the points $B$ and $A'$ respectively, which are both fixed in the moving frame. According to velocity definition,

$$\boldsymbol{v}_{B'\mathrm{e}} - \boldsymbol{v}_{A'\mathrm{f}} = \frac{\mathrm{d}\boldsymbol{r}_{B'}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{r}_{A'}}{\mathrm{d}t} = \frac{\mathrm{d}(\boldsymbol{r}_{B'} - \boldsymbol{r}_{A'})}{\mathrm{d}t} = \frac{\mathrm{d}\overline{A'B'}}{\mathrm{d}t},$$

where $\overline{A'B'}$ is a length-fixed vector inscribed to the moving frame. Once again, Eq. (16) is applied, as a result,

$$\boldsymbol{v}_{B'\mathrm{e}} - \boldsymbol{v}_{A'\mathrm{f}} = \frac{\mathrm{d}\overline{A'B'}}{\mathrm{d}t} = \boldsymbol{\omega} \times \overline{A'B'}. \tag{24}$$

Now Eq. (23) can be reformulated as

$$\lim_{\Delta t \to 0} \frac{\boldsymbol{v}_{B'\mathrm{e}} - \boldsymbol{v}_{A\mathrm{e}}}{\Delta t} = \boldsymbol{\omega} \times \lim_{\Delta t \to 0} \frac{\overline{A'B'}}{\Delta t} + \boldsymbol{a}_{\mathrm{e}}. \tag{25}$$

In the moving frame, the particle move from $A'(A)$ at instant $t$ to $B'$ at instant $t+\Delta t$, so

$$\lim_{\Delta t \to 0} \frac{\overline{A'B'}}{\Delta t} = \boldsymbol{v}_{A\mathrm{r}}.$$

That is Eq. (25) is

$$\lim_{\Delta t \to 0} \frac{\boldsymbol{v}_{B'\mathrm{e}} - \boldsymbol{v}_{A\mathrm{e}}}{\Delta t} = \boldsymbol{\omega} \times \boldsymbol{v}_{A\mathrm{r}} + \boldsymbol{a}_{\mathrm{e}}. \tag{26}$$

Combining Eq. (19), Eq. (20), Eq. (21), and Eq. (26) together leads a singular formula

$$\boldsymbol{a}_{\mathrm{a}} = \boldsymbol{a}_{\mathrm{e}} + \boldsymbol{a}_{\mathrm{r}} + \boldsymbol{a}_{\mathrm{C}}, \tag{27}$$

where $\boldsymbol{a}_{\mathrm{C}}$ is the so-called "Coriolis acceleration". This acceleration is

$$\boldsymbol{a}_{\mathrm{C}} = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}_{A'\mathrm{r}} - \boldsymbol{v}_{A\mathrm{r}}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\boldsymbol{v}_{B'\mathrm{e}} - \boldsymbol{v}_{A'\mathrm{f}}}{\Delta t} = 2\boldsymbol{\omega} \times \boldsymbol{v}_{A\mathrm{r}}. \tag{28}$$

If the moving frame is translated without rotation, then $\boldsymbol{\omega} = \mathbf{0}$. For this case, Eq. (26) is reduced to

$$\boldsymbol{a}_{\mathrm{a}} = \boldsymbol{a}_{\mathrm{e}} + \boldsymbol{a}_{\mathrm{r}}. \tag{29}$$

This is similar to the velocity formula for the moving frame.

Eq. (28) indicates that there are two factors contributing to the Coriolis acceleration. First factor, the direction of the relative velocity viewed from the fixed frame is changed as the moving frame moves. Second factor, the particle moves in the moving frame, which leads to the refresh of the entrained points. With this decomposition, the physical meaning of the Coriolis acceleration is clear.

## V. CONCLUSION

A concise and self-closed way is presented to derive the acceleration formula for a moving frame. This procedure represents a clear physical meaning of Coriolis acceleration.

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