Motional emfs and the Hall effect

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(Received 8 November 2011, accepted 26 December 2011)

Abstract

When a moving conductor shaped like a rod or wire interacts with a static external magnetic field, two effects conventionally associated with the direct action of magnetic forces arise. The first is an induced emf which in typical textbook accounts is initiated and maintained by a force proportional to the velocity of the conductor. The second is a resistive force, proportional to the induced current, presumed to act on the conduction electrons. We present an alternate theory to explain both effects that relies on an electric field within the conductor that has both transverse and axial components. The transverse field is analogous to the electric field associated with the Hall effect. The Hall field acts to transfer energy to the electrons, which generates the emf, and impede the motion of the ions, which is the origin of the resistive force. The combination of the axial field and the magnetic field is shown to act like a velocity selector. This clarifies the role of the magnetic field and avoids confusion about the energy transfer process (*i.e*. that magnetic forces can do mechanical work).

Keywords: Electromagnetism, Physics Education.

Resumen

Cuando un conductor en movimiento forma una varilla o cable interactuando con un campo magnético estático externo, surgen dos efectos convencionalmente asociados con la acción directa de las fuerzas magnéticas. La primera es una fuerza electromotriz inducida que en los libro de texto típicos es iniciada y mantenida por una fuerza proporcional a la velocidad del conductor. La segunda es una fuerza de resistencia proporcional a la corriente inducida, se presume que actúan sobre los electrones de conducción. Se presenta una teoría alternativa para explicar los efectos la cual se basa en un campo eléctrico dentro del conductor que tiene componentes transversal y axial. El campo transversal es análogo al del campo eléctrico asociado con el efecto Hall. El campo de Hall actúa para transferir energía a los electrones, los cuales generan la fem, e impiden el movimiento de los iones, lo cual es el origen de la fuerza resistiva. La combinación del campo axial y del campo magnético se muestra que actúa como un selector de velocidad. Esto aclara el papel del campo magnético y evita la confusión sobre el proceso de transferencia de energía (es decir, que las fuerzas magnéticas pueden hacer trabajo mecánico).

Palabras clave: Electromagnetismo, Educación en Física.

PACS: 41.20.Gz, 45.20.dg, 72.15.Eb **ISSN 1870-9095**

I. INTRODUCTION

It is a well established fact that magnetic forces are incapable of doing mechanical work on charged particles such as electrons due to the fact that they always act perpendicular to the displacement of the particle. Several electromagnetic phenomena, both part of the undergraduate curriculum, involving the interaction of a conductor with an external magnetic field appear to challenge this notion. The first is the behavior of a current carrying wire interacting with an external magnetic field. The second, treated here, is the case of a moving conductor, also interacting with an external magnetic field, that acts as the seat for a motional emf. In each case, students can become confused by the seemingly contradictory explanations that appear in many texts [1, 2]. Instructors know better but may find it difficult to offer consistent alternative explanations**.**

A force proportional to the product of the current and the external magnetic field acts on the conductor in each case. This force accelerates the current carrying conductor and offers resistance, consistent with Lenz's law, when the moving conductor acts as the seat of a motional emf. Both forces are usually said to act on the conduction electrons and are identified as being of magnetic origin. In neither case does the force act perpendicular to the displacement, which suggests that it is doing work. In the case of the current carrying conductor several authors [3,4] have offered alternative explanations that involve an internal (to the conductor) transverse electric field analogous to the field associated with the Hall effect. The electric field acts to accelerate the ions which comprise almost all of the mass of the conductor. Since electric fields can do work their approach seems to resolve the work-energy dilemma for this case. The general intent of this paper is to test the validity of that approach in regard to the motional emf case. Part of the task will be to explain the resistive force in electrostatic terms. But, in addition, we also attempt to resolve work-energy issues associated with the presence of the velocity dependent emf. In particular, exploring the role that the electric field plays in the electron dynamics helps explain how energy is transferred to the electrons from the external agent that exerts a force on the moving conductor.

The organization of the paper is as follows. In sections II and III we describe the system under consideration and develop the basic model which we use to describe the electron and ion motions. We use this model in section IV to show how the resistive force is electrostatic, rather than magnetic, in nature. In section V we expand on this model to include energy considerations and using a velocity selector analogy illustrate the mechanism underlying the emf that develops through the motion of the conductor. In section VI we develop a model of both the Hall field and the transverse charge drift that supports it. Section VII is reserved for discussion and concluding remarks.

II. THE SYSTEM BEING CONSIDERED

A concrete example of the specific phenomena treated here is shown in Fig. 1 where we show an apparatus capable of generating electric current through the motion of a conductor in a region where a magnetic field exists. A thin metal rod (or wire) with uniform circular cross section is able to slide, without friction and rolling, along the surface of a track consisting of a pair of conducting rails. The rod's transverse motion, (center of mass) as viewed by an observer at rest with the remainder of the circuit (lab frame), is parallel to the positive x axis and so we describe its velocity with the vector $\mathbf{v}_{cm} = v_{cm}\hat{\mathbf{x}}$ which can depend on time. A uniform magnetic field $\mathbf{B}_{ext} = \mathrm{B}_{ext}\hat{\mathbf{z}}$ is directed out of the page produced by some external source such as a permanent magnet or coil. An external resistor is connected into what can then be viewed as a simple series circuit. Current I (which can depend on time) flows in the circuit as shown. Within the rod the electron drift is parallel to the positive y axis.

The basic facts are not in dispute. If the instantaneous transverse velocity of the rod is $\mathbf{v}_{cm}$, a difference of potential will develop along the length of the rod driven by a velocity dependent emf given by $LB_{ext}v_{cm}$. An external force $\mathbf{F}_{ext}$ is required to first accelerate the rod, but then to also maintain constant velocity motion as the motion is opposed by a force of magnitude ILB_{ext}. Both the induced difference of potential and the resistive force can be observed in fairly simple experiments [5].

The theory developed in textbooks [1,2] focuses on the role played by a force of magnetic origin proportional to $\mathbf{v}_{cm} \times \mathbf{B}_{ext}$ that acts on the conduction electrons which are assumed to move transversely with the ions. This force then initiates and maintains the emf. The resistive force (ILB_{ext}) is identified as a second magnetic force also acting on the electrons due to their axial drift initiated by the emf.

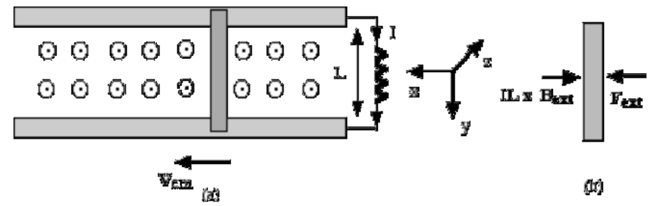

FIGURE 1. (a) An apparatus to generate a motional emf. (b) Two forces act on the moving conductor. One is proportional to the current.

III. THE BASIC MODEL

Let's imagine an external agent exerting a constant contact force on the rod in Fig. 1 (through its center of mass). Since in an ideal sense the ions within are rigidly attached to one another, it seems reasonable to conclude that this force will act directly on the ions in the vicinity of the point of contact and then indirectly through interatomic forces on the remainder. Since the contact force does not act directly on the conduction electrons they will lag behind the now accelerating ions as viewed by an observer in the lab frame. The initially neutral charge distribution now becomes polarized and a transverse electric field will be established across the rod with an orientation similar to what is shown in Fig. 2a. Inertia induced electric fields in conductors have been studied before including the well known experiments of Tolman and Stewart [6, 7].

The field that grows from this inertial seed is analogous to the electric field associated with the Hall effect in metals. We show in section VI that while the rod accelerates the Hall field grows in magnitude through the interaction of the drifting electrons with the external magnetic field, much like it does in a stationary conductor. The Hall field is oriented in such a manner that it can transfer energy to the electrons and do work against, and thus resist, the motion of the ions. To the lab based observer both groups of particles are in motion.

If we view the rod in cross-section then initially the uniform and equal distributions of electrons and ions produce electric fields that cancel at every point within the interior. For a sufficiently long rod we can ignore edge effects and take advantage of the axial symmetry. Then applying Gauss's law we can write $\mathbf{E}_e = \frac{-en}{2\varepsilon_0}\mathbf{r}$ and $\mathbf{E}_i = \frac{en}{2\varepsilon_0}\mathbf{r}$ where $\mathbf{r} = x_c\hat{\mathbf{x}} + z_c\hat{\mathbf{z}}$ as in Fig. 2b to describe the electron and ion fields within the interior. The coordinates refer to the comoving frame. In the lab frame $x(t) = x_c + x_0(t)$ where $x_0(t)$ is the x coordinate of the comoving origin. To the lab based observer the position of the center of mass of the rod is given by: $\{x_0(t), 0, 0\}$. When the wire is at rest the field at position $\mathbf{r}$ in Fig. 2b is zero.

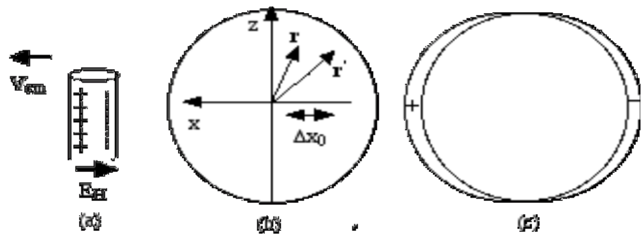

FIGURE 2. (a) The transverse electric field initially develops from the inertial lag of the conduction electrons. (b) The two position vectors shown, within the interior of the rod, have the same z component. If the ion field at the point on the right were to overlap with the electron field at the point on the left due to a shift in the ion distribution, the z components of the field vectors would cancel. (c) When the ion distribution shifts thin surface layers of charge will be exposed on opposite sides of the rod.

The position vector $\mathbf{r}'$ has the same z component as $\mathbf{r}$ and thus the z component of the ion field at the two points is the same. If the ions accelerate and the electrons lag, the ion field at $\mathbf{r}'$ will superimpose with the electron field at $\mathbf{r}$. Let $\mathbf{r}'-\mathbf{r}=-\Delta x_0\hat{\mathbf{x}}$ then $\mathbf{E}_e(\mathbf{r})+\mathbf{E}_i(\mathbf{r}')=\frac{-en}{2\varepsilon_0}\Delta x_0\hat{\mathbf{x}}$.

Thus a shift in the ion distribution to the left by an amount Δx_0 results in a field in the interior directed to the right given by

$$\mathbf{E}_0=\frac{-en}{2\varepsilon_0}\Delta x_0\hat{\mathbf{x}}. \tag{1}$$

This field is uniform as other than having the same z coordinates the two positions chosen are arbitrary.

We treat the inertial field described by Eq. (1) as the initial condition for the electric field within the conductor. We show later that at some threshold value of Δx_0 the electrons will begin to accelerate. Once in motion relative to the lab frame they will interact with the magnetic field. The axial drift of the electrons will then initiate an axial field directed parallel with the positive y axis. Once a current is established both components of the field will grow with time. Thus we write: $\mathbf{E}=-E_H\hat{\mathbf{x}}+E_a\hat{\mathbf{y}}$ for the field within the moving conductor where both field components can depend on time. Ignoring edge effects we expect the electric field to remain uniform given the uniformity of the initial conditions.

We use a classical particle model of the electron and ion motions as viewed in the lab frame. The x components of Newton's second law for each species of particle are shown below:

$$m_i a_x^i = f - eE_H + R_x^i, \tag{2a}$$

$$m_e a_x^e = eE_H - ev_y^e B_{ext} + R_x^e, \tag{2b}$$

where f represents the additional interatomic force that acts on the ion when the external force is applied and v_y^e is the y component of the electron's velocity. A force term appears in each equation to represent collisions. These are random forces, however, their mean values can be represented in a phenomenological manner using a force law that resembles velocity dependent drag [4]. For an observer at rest with the ions we describe the collision force that acts on an electron using $\langle R_x^e\rangle=\frac{-m_e}{\tau_e}\langle v_x^e\rangle$ where the relaxation time, τ_e, is the mean free time. The brackets refer to mean instantaneous values averaged over all the conduction electrons (or ions as the case may be) in the rod. In the lab frame we must consider a Galilean transformation of the above force law:

$$\langle R_x^e\rangle=\frac{-m_e}{\tau_e}\left[\langle v_x^e\rangle-\langle v_x^i\rangle\right]. \tag{3}$$

The mean electron and ion forces are related by Newton's third law: $\langle R_x^e\rangle+\langle R_x^i\rangle=0$ ignoring electron-electron collisions.

IV. THE NATURE OF THE RESISTIVE FORCE

We can use Eqs. (2a) and (2b) to account for the resistive force, which is the force that directly opposes the external force. The most straightforward measurement of the resistive force would occur after the rod has reached terminal velocity when the mean electron and ion velocities are equal. Once this steady state is reached we have $\langle a_x^i\rangle=\langle a_x^e\rangle=0$ and $\langle R_x^e\rangle=\langle R_x^i\rangle=0$. Applying these conditions, we then average Eqs. (2a) and (2b) assuming a spatially uniform Hall field as discussed above

$$\langle f\rangle = eE_H\ , \tag{4a}$$

$$B_{ext}\langle v_y^e\rangle = E_H. \tag{4b}$$

There are $N=nLA$ ions in the rod, where n is the number density and A is the cross-section, so the total force that acts on the ion lattice by Eq. (4a) is

$$F_{ext}=nLA\langle f\rangle = nLAeE_H. \tag{5}$$

In the steady state the Hall field depends on the electron drift as it would in a stationary conductor according to Eq. (4b). $\langle v_y^e\rangle$ is the drift velocity of the electrons. As a result the net force that opposes the ion motion is proportional to the current. To show this we substitute Eq. (4b) into Eq. (5). Then, since the current in the rod is given by $I=en\langle v_y\rangle A$, we have

$$F_{ext}=ILB_{ext}. \tag{6}$$

We are assuming for simplicity that the ion and electron number densities are equal, which is commonly the case in metals.

In calculating the current we determine the flux of the vector $j_y\hat{\mathbf{y}}$ through the cross-section of the rod, where $j_y = -en\left\langle v_y^e \right\rangle$ and we define the vector $\mathbf{A} = -A\hat{\mathbf{y}}$ and write $I = j_y\hat{\mathbf{y}} \bullet \mathbf{A}$. This reproduces the output of an ammeter connected in series with the moving rod. We use the same definition in calculations that follow.

The right hand side of Eq. (6) represents the resistive force observed which directly opposes $\mathbf{F}_{ext}$. It is an electrostatic force that does work against the motion of the ions. The ions comprise greater than 99.9% of the mass of most metals and can be approximately modeled as a rigid body. In purely mechanical terms the external force must accelerate and maintain their motion. The presence of an inertial field, however, means that the resistive force is not generally proportional to the current, only a linear function of it.

The magnetic force acts directly on the electrons only. The resultant force that resists their motion also has magnitude ILB_{ext} as can be seen if we multiply Eq. (4b) by *N*. A net electrostatic force of the same magnitude acts in the direction of their mean transverse motion. The force that opposes the electron motion is of magnetic origin, but it is important to realize that ILB_{ext} is only a component of the magnetic force. The magnetic force is not doing work. We expand on this point in the next section.

V. THE INDUCED EMF

The term electromotive force or emf generally is used to describe a process or agent of non-electrostatic origin through which charge carriers acquire enough energy to overcome the effects of an electrostatic field that opposes their motion. Most commonly the term is used in connection with a battery. In that context one is generally talking about an exothermic chemical reaction, involving the battery terminals, that transfers charge to the terminals from an electrolyte. Energy is required because one outcome of the accumulation of charge on the terminals is an electrostatic field that opposes the charge drift. Especially from a pedagogical point of view the process would seem more concrete if some field were present to oppose the electrostatic field. Unfortunately this is not the case. The charge carriers simply gain enough kinetic energy via the chemical reaction to climb the potential hill surrounding each electrode.

The conventional explanation of motional emf's relies on the existence of a field (so to speak) described by $\mathbf{v_{cm}} \times \mathbf{B_{ext}}$ which forces charged particles to drift against the electrostatic repulsion that results from the accumulation of charge at either end of the conductor. In contrast, the model we develop here is more like a battery. The non-electrostatic influence is the force exerted on the conductor by the external agent. As discussed above this force acts principally on the ions. Energy is transferred to the system through the work done by this force, and then to the electrons through work done by another transverse force; the electrostatic force exerted by the Hall field. No work doing force directly opposes the axial electric field, but rather the electrons acquire enough kinetic energy through their interaction with the Hall field to overcome the potential gradient. In order for the Hall field to do work transverse displacement is required, hence the requirement of motion relative to the lab frame. The magnetic force that acts on the electrons also relies, in part, on transverse electron motion. However, in our model its role is to steer the electron motion along the axis of the rod without doing any work. It therefore changes the momentum of the electrons without affecting their energy.

By analogy the system behaves like a velocity selector in that we have a pair of crossed fields, the axial *E* field and the external *B* field, that form the selector and an accelerating electric field, the Hall field, that injects particles into the selector. This can be seen if we develop a model for the axial drift of the electrons, namely

$$\left\langle v_y^e \right\rangle = \frac{e\tau_e}{m_e}\left[\left\langle v_x^e \right\rangle B_{ext} - E_a\right] = \frac{e\tau_e}{m_e}B_{ext}\left[\left\langle v_x^e \right\rangle - V_{sx}\right] \tag{7}$$

where $V_{sx} = \dfrac{E_a}{B_{ext}}$ is the x component of the selector velocity. Thus according to Eq. (7) there is axial drift of electrons (and therefore current) as long as the mean transverse velocity exceeds the selector velocity. This means that during the initial transient when the rod is accelerating, and the axial field is small, the current rises. In a closed circuit once the rod reaches terminal speed it is the discharge at the rod's ends that lowers the axial field and allows for current flow within the rod.

To model the energy transfer (see Fig. 3) we calculate the mean instantateous rate at which a conduction electron's kinetic energy changes from the equation below

$$\left\langle \frac{dK^e}{dt} \right\rangle = \left\langle \mathbf{F} \bullet \mathbf{v_e} \right\rangle. \tag{8}$$

$\mathbf{v}_e = v_x^e\hat{\mathbf{x}} + v_y^e\hat{\mathbf{y}}$ and $\mathbf{F} = -e\mathbf{E} + \mathbf{F}_B + R_x^e\hat{\mathbf{x}} + R_y^e\hat{\mathbf{y}}$ which includes the electrostatic force, the magnetic force defined by $\mathbf{F}_B = -ev_y^e B_{ext}\hat{\mathbf{x}} + ev_x^e B_{ext}\hat{\mathbf{y}}$ and the effects of collisions where we have used Eq. (3) and in a similar manner write $\left\langle R_y^e \right\rangle = \dfrac{-m_e}{\tau_e}\left\langle v_y^e \right\rangle$. It is clear that the product $\mathbf{F}_B \bullet \mathbf{v}_e = 0$ and so no work is done by the magnetic force. The mean value of the y component of this force is the $\mathbf{v}_{cm} \times \mathbf{B}_{ext}$ force and the x component (multiplied by N) the ILB_{ext} force. The work done by each component exactly cancels the other.

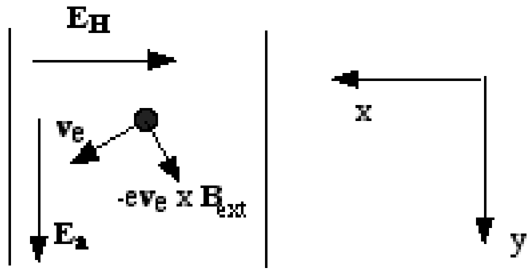

FIGURE 3. A typical electron interacts with the transverse and axial electric fields as well as the magnetic field.

To fully understand the phenomena we have not made any assumptions in regard to the relative electron-ion motions. Thus the collision terms associated with the transverse motion have been included. Transverse electron drift can be described by the quantity $v_x^d = \langle v_x^e \rangle - \langle v_x^i \rangle$ which is the x component of the drift velocity. We then calculate the x and y components of the current density as follows:

$$j_x = -nev_x^d = -ne\left[\langle v_x^e \rangle - \langle v_x^i \rangle\right] \quad (9a)$$

$$j_y = -nev_y^d = -ne\langle v_y^e \rangle . \quad (9b)$$

Then from Eqs. (8), (9a) and (9b) we find

$$\left\langle \frac{dK^e}{dt} \right\rangle + \frac{\rho}{n} j^2 = eE_H \langle v_x^e \rangle - eE_a \langle v_y^e \rangle + \langle R_x^e \rangle \langle v_x^i \rangle \quad (10)$$

where $j^2 = j_x^2 + j_y^2$ and $\rho = \frac{m_e}{ne^2 \tau_e}$ is the resistivity.

The Joule heating term on the left includes both axial and transverse contributions. The last term on the right would appear to represent an inertial effect which is present to the lab based observer for whom both the ions and electrons are in motion, though not necessarily together. If the electrons lag the ions, for example, this term would lead to an energy gain by the electrons.

There is no observed gain in kinetic energy associated with the electron motion in the y direction. All the energy gain appears as Joule heating. Thus we find that

$$\left\langle \frac{dK^e}{dt} \right\rangle = \left\langle \frac{d}{dt} \frac{1}{2} m_e v_x^{e2} \right\rangle = \left\langle v_x^e m_e \frac{d}{dt} v_x^e \right\rangle ,$$

then using Eq. (2b)

$$\left\langle \frac{dK^e}{dt} \right\rangle = eE_H \langle v_x^e \rangle - e\langle v_y^e \rangle B_{ext} \langle v_x^e \rangle - \frac{\rho}{n} j_x^2 + \langle R_x^e \rangle \langle v_x^i \rangle . \quad (11)$$

Substituting Eq. (11) into Eq. (10) and cancelling terms we find

$$\rho j_y^2 + \left[B_{ext} \langle v_x^e \rangle - E_a \right] j_y = 0 . \quad (12)$$

If we multiply Eq. (12) by N we can cast it into a more familiar form after noting that $\Delta V_{rod} = E_a L$ is the difference of potential across the rod, $I = -j_y A$ is the current, and $r = \rho \frac{L}{A}$ the internal resistance of the rod:

$$I\Delta V_{rod} = IB_{ext} L \langle v_x^e \rangle - I^2 r . \quad (13)$$

Using the loop rule we can write $\Delta V_{rod} = IR_{ext}$, where R_{ext} is the equivalent resistance of the rails and external resistor. Then

$$I^2 R_{eq} = IB_{ext} L \langle v_x^e \rangle , \quad (14)$$

where $R_{eq} = R_{ext} + r$. In both Eq. (13) and Eq. (14) the term $\mathrm{E} = B_{ext} L \langle v_x^e \rangle$ can be identified as the emf. This is the standard result found in textbooks. It depends on the mean electron velocity (x component) which is usually assumed to be equivalent to the center of mass velocity of the conductor. Our theory then reproduces the accepted results that are usually arrived at without reference to a transverse electric field.. The standard explanation, however, is vague in regard to energy transfer. We believe the process is much clearer using the model developed here. For example, if we multiply Eq. (11) by N and for the moment ignore the terms dependent on transverse drift we can understand how the power input via the Hall field, $P_H = NeE_H \langle v_x^e \rangle$ both increases the kinetic energy of the electron center of mass, $K_{cm}^e = N\langle K^e \rangle$, and powers the emf:

$$\frac{dK_{cm}^e}{dt} + IB_{ext} L \langle v_x^e \rangle = P_H , \quad (15)$$

or using Eq. (14) generates internal energy through Joule heating. A statement of conservation of energy for the whole system can be obtained in a similar manner by using Eq. (2a) to bring the ions into the system described by Eq. (10). We find from (2a) that

$$\left\langle \frac{d}{dt} K^i \right\rangle = \langle f \rangle \langle v_x^i \rangle - eE_H \langle v_x^i \rangle - \langle R_x^e \rangle \langle v_x^i \rangle$$

where we have used the condition $\langle R_x^e \rangle + \langle R_x^i \rangle = 0$.

Then adding this result to Eq. (10) and multiplying by *N* we find

$$\frac{d}{dt} K_{system} + I^2 R_{eq} + LA\left[E_H j_x + \rho j_x^2 \right] = F_{ext} \langle v_x^i \rangle , \quad (16)$$

where $\frac{d}{dt}K_{system} = N\left[\left\langle \frac{d}{dt}K^e \right\rangle + \left\langle \frac{d}{dt}K^i \right\rangle\right]$ and we have used the loop rule. This expression includes two terms that depend on the transverse current. One describes the energy input needed for electrons to drift against the Hall field while the field grows, and the second the Joule heating that will also occur during this drift. We show in the next section that the transverse current is very small and as a result these terms will be negligible. However, they describe processes that must be present. Once a steady state is reached the transverse current vanishes and all of the work done by the external agent appears as Joule heating associated with the axial current density.

VI. THE TRANSVERSE FIELD

In section III we discussed how the Hall field could arise through the inertia of the conduction electrons. In this section we develop this idea in more detail and also develop a model to understand the dependence of the field on the current.

As the rod accelerates the Hall field grows with time. The direct dependence is actually on current much like in a stationary conductor. If we examine particular cases we can see this dependence explicitly. For example, using Eqs. (2a) and (2b) and assuming that the mean electron and ion accelerations are equal we find that the Hall field must satisfy

$$eE_H = N^{-1}\left[\frac{\mu}{m_i}F_{ext} + \frac{\mu}{m_e}ILB_{ext}\right]. \qquad (17)$$

where $\mu = \frac{m_i m_e}{m_i + m_e}$ is the reduced mass of an electron-ion pair.

The current dependence for this case is clear. The first term on the right in Eq. (17) represents, if divided by e, the magnitude of the Hall field for zero current. This then must be the field created by the inertial lag of the electrons. Note that since the initial acceleration of the ions is $a_0 = \frac{\langle f \rangle}{m_i}$, the zero current term is just $\mu a_0 = \frac{m_i}{m_i + m_e} m_e a_0$. Thus it is slightly less than the initial "ma" force that would appear to act on an electron in the non-inertial frame of the accelerating rod. The mass factor $\frac{\mu}{m_e} = \frac{m_i}{m_i + m_e}$ is approximately equal to unity as the typical ion mass is so much larger than the electron mass.

We can then equate this inertial term with Eq. (1) to determine the displacement Δx_0. A little algebra yields

$$\frac{F_{ext}}{M} = \frac{\sigma}{2\tau_e \varepsilon_0}\Delta x_0,$$

where M is the mass of the rod and $\sigma = \frac{ne^2\tau_e}{m_e}$ is the conductivity. For a copper rod ($n \approx 8.5\times10^{28} m^{-3}$, $\sigma \approx 5.95\times10^7 (\Omega \bullet m)^{-1}$ density $\approx 8.9\times10^3\ kg/m^3$) 50cm in length and 1 mm in diameter with $F_{ext} = 0.1N$ we find that the inertial piece of the field requires a very small displacement: $\Delta x_0 \approx 2.1\times10^{-31} m$. If the initial charge distribution is the seed from which the field grows and a field of greater magnitude requires an expansion of the initial distribution then Eq. (1) should still apply. According to Eq. (2a) when the rod reaches terminal velocity the Hall field has magnitude $\frac{F_{ext}}{Ne}$. If we again use Eq. (1) for the same rod subject to the same force we find (dropping the subscript) $\Delta x \approx 4.9\times10^{-25} m$ which is still very small but is comparable to a similar result obtained by other authors [3] in their analysis of the force on a current carrying conductor. Regardless, it would appear that a Hall field of sufficient magnitude requires only a modification to the surface charge distribution on the conductor. To be more specific, given the small thickness of each of the slivers in Fig. 2 at their widest points we represent the volume of each by $\Delta V \approx \frac{\pi}{2} d\Delta x L$. The fraction of the total volume occupied by each sliver is then $\frac{\Delta V}{V} \approx 2\frac{\Delta x}{d} = 9.8\times10^{-22}$ using the steady state field. Given that there are 1.67×10^{22} electrons within the copper rod being discussed we must conclude that this is a surface charge. It would appear then, that like the axial field [8], the appearance of a transverse field requires, for the most part, a modification of the surface charge density.

The Hall field is both initiated and expanded in magnitude through the lag of the electrons relative to the ions. This lag (or drift) is either inertial or directed through interaction with the external magnetic field. At best we are talking about a small deviation in the otherwise axially directed current density. However, since the growth of the Hall field is an important component of the feedback mechanism that causes the rod to reach terminal velocity (see Eq. (2a)), transverse electron drift must also play a role. This is an especially important issue for teaching purposes as many students need some sense of mechanism in order to understand a process.

To understand how the x component of the drift velocity depends on the Hall field and other variables we determine the quantity $d\Delta/dt$ where $\Delta = \langle v_x^e \rangle - \langle v_x^i \rangle$ by using Eqs. (2a) and (2b):

$$\frac{d}{dt}\Delta = \frac{e}{\mu}E_H - \frac{\langle f \rangle}{m_i} - \frac{e}{m_e}\langle v_y^e \rangle B_{ext} - \tau_e^{-1}\Delta .$$

To arrive at this result we have dropped a term multiplied by the factor m_e/m_i in the ion equation.

Both the Hall field and the axial drift velocity change over a time scale much larger than the mean free time. Over a time interval comparable to the mean free time they can be treated as constants. Under these conditions $d\Delta/dt$ decays like $\exp(-t/\tau_e)$. We then equate the terminal value of Δ, obtained by holding E_H and $\langle v_y^e \rangle$ constant and letting $d\Delta/dt \to 0$, with the transverse drift velocity. Then we use Eq. (9a) and some algebra to find

$$j_x = -\frac{m_e}{\mu}\sigma\left[E_H - \frac{\mu}{m_i}\frac{\langle f\rangle}{e}\right] - \frac{\sigma B_{ext}}{ne} j_y . \qquad (18)$$

According to Eq. (14) since $I = -j_y A$ we can re-write Eq. (18) in terms of the mean transverse electron velocity. However, the directly observable velocity is the center of mass velocity of the moving rod: $V_{cm} = \frac{\mu}{m_i}\langle v_x^e \rangle + \frac{\mu}{m_e}\langle v_x^i \rangle$.

We can then use Eq. (9a) to find that: $\langle v_x^e \rangle = V_{cm} - \frac{\mu}{m_e}\frac{j_x}{ne}$.

Which means

$$j_x = -\varepsilon_0 \gamma\left[E_H - \frac{\mu}{m_i}\frac{\langle f\rangle}{e}\right] + \frac{\mu}{m_e}\left(\frac{ne\Omega}{\sigma}\right)\varepsilon_0 \gamma V_{cm} , \qquad (19)$$

where $\Omega = \left(\frac{\sigma B_{ext}}{ne}\right)^2 \frac{r}{R_{eq}}$ and $\gamma = \frac{m_e}{\mu}\frac{\sigma}{\varepsilon_0}\left(1 + \frac{\mu}{m_e}\Omega\right)^{-1}$. The dimensionless constant Ω is very small for the copper rod discussed previously. Namely, if $R_{ext} = 1.0$ Ohm, $\Omega \approx 4\times 10^{-9}$.

The transverse current density is uniform according to (19) within the interior of the rod. The actual transport of charge into the surface region will not be uniform as the current (the flux of the current density) will decrease for points away from the x axis. Thus the accumulation of charge will be regulated by the geometry. For this reason we expect the charge accumulation to preserve the initial geometry. Since the source charges for the Hall field appear to be confined to the surface, and both components of the field are uniform within the interior, $\nabla \bullet \mathbf{E}$ will vanish everywhere except at the surface.

The x axis in Fig. 2 passes through the surface charge region at its widest point. The boundary between the interior and surface regions along the negative x axis is located at $x = -d + \Delta x$, in the commoving frame, where Δx is the thickness. The Hall field is normal to the boundary at this point. Since the axial field component is tangential to the surface, we can write $\nabla \bullet \mathbf{E} = \hat{\mathbf{x}}\frac{\partial}{\partial x} \bullet \left(-E_H \hat{\mathbf{x}}\right) = -\frac{\partial E_H}{\partial x}$. Then, if η is the charge density on the other side of the boundary, by Gauss's law we write

$$-\frac{\partial E_H}{\partial x} = \frac{\eta}{\varepsilon_0} . \qquad (20)$$

But since charge is conserved we can also write $\frac{\partial \eta}{\partial t} + \frac{\partial j_x}{\partial x} = 0$. Then from Eq. (20) we have

$$\varepsilon_0 \frac{\partial^2 E_H}{\partial t\, \partial x} - \frac{\partial j_x}{\partial x} = 0 ,$$

which we integrate to find

$$\frac{\partial E_H}{\partial t} - \frac{j_x}{\varepsilon_0} = 0 . \qquad (21)$$

Then substituting Eq. (19) into Eq. (21) and dropping the partial derivative notation we find

$$\frac{d}{dt}E_H + \gamma\left[E_H - \frac{\mu}{m_i}\frac{\langle f\rangle}{e}\right] = \frac{\mu}{m_e}\frac{ne\Omega}{\sigma}\gamma V_{cm} , \qquad (22)$$

which describes Hall field within the interior region.

Eq. (22) can be solved using a Green's function approach

$$E_H(t) = \frac{\mu}{m_i}\frac{\langle f\rangle}{e} + \frac{\mu}{m_e}\frac{ne\Omega}{\sigma}\gamma\int_0^t dt' \exp\left(-\gamma(t-t')\right)V_{cm}(t') . \qquad (23)$$

To analyze the effect of an applied constant force we add Eqs. (2a) and (2b) to determine the equation of motion for the center of mass of the system:

$$\frac{d}{dt}V_{cm} = M^{-1}\left[F_{ext} - ILB_{ext}\right],$$

where *M* is the mass of the rod. This equation has solution:

$$V_{cm}(t) = \frac{F_{ext}\tau}{M}\left[1 - \exp(-t/\tau)\right], \qquad (24)$$

where $\tau = \frac{MR_{eq}}{(LB_{ext})^2}$ if we use Eq. (14) and let $V_{cm} \approx \langle v_x^e \rangle$ thus assuming that the transverse drift has very little effect on the center of mass motion. We have also neglected a very small initial velocity whose effects would decay like $\exp(-t/\tau)$. The ions are not at rest at the point that the field starts to increase through current flow, but their velocity is extremely small. A simple model where we assume a constant acceleration $a_0 = \frac{\langle f\rangle}{m_i}$ yielding a displacement Δx_0 as in Eq. (1) results in an initial velocity

$v_0 \approx \sqrt{2a_0\Delta x_0}$ ($\approx 3.5\times10^{-15}\, m/s$ for the copper rod discussed).

After substituting Eq. (24) into Eq. (23), performing the integration in Eq. (23) and some algebra we find

$$E_H(t) = \frac{\mu}{m_i}E_\infty + \frac{\mu}{m_e}E_\infty\left[1 - \frac{\gamma}{\gamma - \tau^{-1}}\exp(-t/\tau)\right] + \frac{\mu}{m_e}\frac{E_\infty}{\gamma\tau}\exp(-\gamma t), \qquad (25)$$

where $E_\infty = \frac{F_{ext}}{Ne} = \frac{\langle f \rangle}{e}$, and we have used the approximation

$$\frac{\gamma}{\gamma - \tau^{-1}} - 1 = \left(1 - (\gamma\tau)^{-1}\right)^{-1} - 1 \approx 1 + (\gamma\tau)^{-1} - 1 = (\gamma\tau)^{-1},$$

to arrive at the term proportional to $\exp(-\gamma t)$. The initial condition in Eq. (25) is the inertial field that appears in Eq. (17). Note that the transverse current described by Eq. (18) does not flow until the axial current is non-zero. It does not describe the short interval during which the inertial field arises.

For realistic applications such as the copper rod discussed above $\frac{\gamma}{\gamma - \tau^{-1}} \approx 1$ as $\gamma >> \tau^{-1}$. As a result, for $t >> \gamma^{-1}$ the Hall field increases over a time scale described by τ and Eq. (25) can be approximated very well by

$$E_H(t) \approx E_\infty\left[1 - \frac{\mu}{m_e}\exp(-t/\tau)\right]. \qquad (26)$$

This is the same result that could be obtained by letting $V_{cm} \approx \langle v_x^e \rangle$ substituting Eq. (24) into Eq. (14) and then substituting this result into Eq. (17). The factor $\frac{\mu}{m_e} \approx 1$ in Eq. (26) is a result of the Hall field having a non-zero initial value.

If we substitute Eqs. (24) and (25) into Eq. (19) we can solve for the transverse current density as a function of time:

$$j_x \approx \frac{\sigma E_\infty}{\gamma\tau}\exp(-t/\tau) - \frac{\sigma E_\infty}{\gamma\tau}\exp(-\gamma t). \qquad (27)$$

We have used the approximation $\varepsilon_0\gamma \approx \frac{m_e}{\mu}\sigma$. Two very distinct time scales are represented in Eq. (27). For this reason we have expressed the maximum values in the form $\frac{\sigma E_\infty}{\gamma\tau}$ instead of the equivalent $\frac{\varepsilon_0 E_\infty}{\tau}$. $\gamma \approx \frac{\sigma}{\varepsilon_0}$ is approximately $7x10^{18}s^{-1}$ for copper and $\tau \approx 1.4s$ for the copper rod discussed previously. Thus for $t << \tau$ we can approximate Eq. (27) by $j_x \approx \frac{\sigma E_\infty}{\gamma\tau}[1 - \exp(-\gamma t)]$ which describes a rise in the transverse current density against what presumably is the resistance of the inertial field. The term in Eq. (25) proportional to $\exp(-\gamma t)$ describes a piece of the field that fades over the same time scale as current flow is initiated and the longer term growth (of the field) takes hold. It appears to be related to the rise in the transverse current.

In a stationary conductor the transverse field is initially zero, so the transverse current starts at a maximum value and decays over a time scale described by γ^{-1}. The decay in this case occurs over the longer time scale described by τ as can be seen by letting $t >> \gamma^{-1}$ in Eq. (27):

$$j_x \approx \frac{\sigma E_\infty}{\gamma\tau}\exp(-t/\tau). \qquad (28)$$

This is the result of the response of the conductor to the continual rise in the current during the accelerated phase of the motion. In the stationary conductor the same analysis as above yields

$$j_x = \frac{\sigma B_{ext} j_y}{ne}\exp(-\gamma t), \qquad (29)$$

which describes the response to a steady axial current (that appears at t=0). Since in the moving conductor we can write $E_\infty = \frac{B_{ext} j_\infty}{ne}$ where j_∞ is the steady state axial current density, the transformation from Eq. (29) to Eq. (28) involves only a time scale shift $t \to (\gamma\tau)t$.

VII. DISCUSSION

The analysis of the induced emf and resistive force that appear when a moving conductor interacts with an external magnetic field, can leave the mistaken impression that magnetic forces are doing mechanical work. We have shown that including the effects of the transverse component of the electric field within the conductor in this analysis leads to a self consistent theory where the roles of the electric and magnetic forces can be clearly delineated. The magnetic field only affects the momentum of the electrons. Energy transfers involving the ions and the electrons are mediated by the electric field resulting in a current dependent resistive force that acts on the ions and a velocity dependent emf. The first result is consistent with results obtained by others [3]. It would appear that forces commonly referred to as magnetic in origin are actually electrostatic. We note that it may be possible to test this conclusion experimentally, as according to (17) the Hall field slightly leads the current due to the presence of the inertially induced field. In contrast, a resistive force of magnetic origin will be proportional to the current.

We have included in our analysis more mathematical detail then would be needed to present this topic to undergraduates. In particular, the effects of transverse drift are small and could be neglected quantitatively. They should be addressed qualitatively however, in our view, so that students can visualize the processes taking place. Through this analysis we have shown that when the Hall effect occurs in a moving conductor it differs from the same effect in a stationary conductor in two respects. First, the time scale is much longer as the conductor is continually responding to the changing velocity of the conductor. Second, the field is initiated through inertia. From a pedagogical point of view this second feature is an excellent application of Newton's first law. Inertial effects such as this are often overlooked by students [9].

Finally we note that force is generally regarded as a more intuitive concept than energy, and at first glance the phenomena at hand tends to be biased towards forces. For example, by Eq, (7) the axial field reaches an equilibrium value of $\langle v_x^e \rangle B_{ext} = V_{cm} B_{ext}$ while the Hall field according to Eq. (2b) is much weaker: $\langle v_y^e \rangle B_{ext}$. But according to Eq. (10) the work done on an electron by each field component in the steady state will be nearly equal. If not for energy considerations it would be easy to ignore the Hall field in favor of the much larger axial forces (*i.e.* the $\mathbf{V_{cm}} \times \mathbf{B_{ext}}$ force).

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