An English Translation of Bertrand’s Theorem

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Abstract
A beautiful theorem due to J. L. F. Bertrand concerning the laws of attraction that admit bounded closed orbits for arbitrarily chosen initial conditions is translated from French into English.

Keywords: Classical Mechanics, Orbits.

Resumen
Un hermoso teorema debido a la J. L. F. Bertrand sobre las leyes de la atracción que admiten limitadas órbitas cerradas arbitrariamente escogidas para las condiciones iniciales es traducido del francés al inglés.

Palabras clave: Mecánica Clásica, Órbitas.

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I. INTRODUCTION

In 1873, Joseph Louis François Bertrand (1822-1900) [1] published a short but important paper in which he proved that there are two central fields only for which all bounded orbits are closed, namely, Newton’s universal gravitation law, that Bertrand calls ‘la loi de la nature’ (the law of nature) and the isotropic harmonic oscillator law. Because of this additional symmetry it is no wonder that the most essential properties of these two fields were studied by Newton himself who discusses them in the Proposition X and in the Proposition XI of Book I of his *Principia* [2]. Newton shows that both fields give rise to an elliptical orbit with the difference that in the first case the force is directed towards the geometrical centre of the ellipse and in the second case the force is directed to one of the foci. Bertrand’s paper appeared in the *Comptes Rendus* of the Académie des Sciences de Paris where the *mémoires* and communications of the members and correspondent members of that French science academy were published. The academic session at which Bertrand presented his paper took place on Monday 20th October 1873 under the presidency of Mr. de Quatrefages. Bertrand’s result, also known as Bertrand’s theorem, continues to fascinate old and new generations of physicists interested in classical mechanics and unsurprisingly papers devoted to it continue to be produced and published. Bertrand’s proof is concise and elegant and contrary to what one may be led to think by a number of perturbative demonstrations that can be found in modern literature, textbooks and papers on the subject, is fully non-perturbative. Giving the continual interest on this theorem and the fact that English is the *lingua franca* of modern science, the present authors think that an English version of Bertrand’s original proof may be of some value.

II. THE TRANSLATION: ANALYTICAL MECHANICS. — A theorem relative to the motion of a point pulled towards a fixed centre; by Mr. J. Bertrand

The planetary orbits are closed curves; this is the main cause of the stability of our system, and this important circumstance stems from the law of attraction which, whichever the initial circumstances, makes each celestial body not expelled from our system to follow the circumference of an ellipse. Until now it was not observed that the Newtonian law of attraction is the only one that fulfills this condition.

Among the laws of attraction that assume to be null the action at an infinite distance, that of Nature is the only one for which a body *arbitrarily* launched with a speed less that a certain limit and pulled towards a fixed centre, describes necessarily a closed curve about this centre. All laws of attraction *allow* closed orbits, but only the law of Nature *imposes* them.

We prove this theorem in the following way: Let $\varphi(r)$ be the attraction exerted at a distance r on the molecule[1] and directed to the centre of the attraction that we will take as the origin of the coordinates. Denoting by r e θ the two

[1] In the original French manuscript, molecule. Bertrand is certainly referring to a particle.

polar coordinates of the mobile body, we have by virtue of a well-known formula,

$$\varphi(r) = \frac{k^2}{r^2}\left(\frac{1}{r} + \frac{d^2\frac{1}{r}}{d\theta^2}\right),$$

and, setting $\frac{1}{r} = z$,

$$r^2\varphi(r) = \psi(z), \tag{1}$$

$$\frac{d^2z}{d\theta^2} + z - \frac{1}{k^2}\psi(z) = 0\,,$$

Let us multiply both members by $2dz$ and integrate. Setting

$$2\int \psi(z)dz = \varpi(z), \tag{2}$$

we will have

$$\left(\frac{dz}{d\theta}\right)^2 + z^2 - \frac{1}{k^2}\varpi(z) - h = 0\,,$$

h being a constant.

From this one deduces that

$$d\theta = \pm\frac{dz}{\sqrt{h + \frac{1}{k^2}\varpi(z) - z^2}}\,.$$

If the curve represented by the equation that ties *z* to θ is closed, the value of *z* will have maxima and minima for which $dz/d\theta$ will be null and for them the correspondent vector radii normal to the trajectory will be necessarily axes of symmetry. Now, when a curve admits two[2] axes of symmetry, the necessary and sufficient condition for it to be closed is that the angle between them be commensurate with π. Therefore, if α and β represent [respectively] a minimum of *z* and the following maximum, the condition required is expressed by the equation

$$m\pi = \int_\alpha^\beta \frac{dz}{\sqrt{h + \frac{1}{k^2}\varpi(z) - z^2}}\,, \tag{3}$$

where *m* denotes a commensurate number. This equation must hold whichever *h* and *k* might be, consequently, [whichever] the limits α and β that depend on them.

One has

$$h + \frac{1}{k^2}\varpi(\alpha) - \alpha^2 = 0,$$
$$h + \frac{1}{k^2}\varpi(\beta) - \beta^2 = 0,$$

consequently

$$\frac{1}{k^2} = \frac{\beta^2 - \alpha^2}{\tilde{\omega}(\beta) - \tilde{\omega}(\alpha)},$$
$$h = \frac{\alpha^2\tilde{\omega}(\beta) - \beta^2\tilde{\omega}(\alpha)}{\tilde{\omega}(\beta) - \tilde{\omega}(\alpha)},$$

and Eq. (3) becomes

$$m\pi = \int_\alpha^\beta \frac{dz\sqrt{\tilde{\omega}(\beta) - \tilde{\omega}(\alpha)}}{\sqrt{\alpha^2\tilde{\omega}(\beta) - \beta^2\tilde{\omega}(\alpha) + \left(\beta^2 - \alpha^2\right)\tilde{\omega}(z) - z^3\left[\tilde{\omega}(\beta) - \tilde{\omega}(\alpha)\right]}}. \tag{4}$$

The function $\varpi(z)$ must be such that this equation holds for all values of α and β. Moreover, the commensurate number *m* must be constant, for if it varies from one orbit to another one, an infinitely small variation on the initial conditions would imply a finite variation of the number and disposition of the axes of symmetry of the trajectory.

Let us assume that α and β differ infinitesimally [from each other]; let

$$\beta = \alpha + u\,,$$

z between α and β, we can set

$$z = \alpha + \gamma\,,$$

and γ will be, just as *u*, infinitesimally small. Neglecting the infinitesimally small [quantities] of second order we will have

$$\sqrt{\tilde{\omega}(\beta) - \tilde{\omega}(\alpha)} = \sqrt{u\tilde{\omega}'(\alpha)}\,.$$

In the expression under the radical sign in the denominator of the integral (4) the infinitesimally small [quantities] of first order reduce to zero, and the same happens with those of second [order]; it is those of third [order] that are necessary to keep, and neglecting the infinitesimally small [quantities] of fourth order one has

$$\alpha^2\tilde{\omega}(\beta) - \beta^2\tilde{\omega}(\alpha) + \left(\beta^2 - \alpha^2\right)\tilde{\omega}(z) - z^3\left[\tilde{\omega}(\beta) - \tilde{\omega}(\alpha)\right] = \left[\tilde{\omega}'(\alpha) - \alpha\tilde{\omega}''(\alpha)\right]\left(u^2\gamma - u\gamma^2\right).$$

Eq. (4) becomes

$$m\pi = \int_\alpha^\beta \frac{d\gamma\sqrt{\tilde{\omega}'(\alpha)}}{\sqrt{\tilde{\omega}'(\alpha) - \alpha\tilde{\omega}''(\alpha)}\sqrt{u\gamma - \gamma^2}}\,,$$

[2] Certainly, Bertrand means the axes of symmetry, not only two of them.

that is, performing the integration and suppressing common factors

$$m=\sqrt{\frac{\tilde{\omega}'(\alpha)}{\tilde{\omega}'(\alpha)-\alpha\tilde{\omega}''(\alpha)}},$$

or

$$\left(1-m^2\right)\tilde{\omega}'(\alpha)+m^2\tilde{\omega}''(\alpha)=0.$$

From this one deduces that

$$\tilde{\omega}'(\alpha)=\frac{A}{\alpha^{1/m^2-1}},$$

$$\tilde{\omega}(\alpha)=A\frac{\alpha^{2-1/m^2}}{2-\frac{1}{m^2}}+B,$$

A and *B* denoting constants.

From the assumed relations between the functions ϖ, ψ, and φ it follows that

$$\psi(z)=\frac{A}{2z^{1/m^2-1}},$$

$$\varphi(z)=\frac{A}{2}r^{1/m^2-3}.$$

Such is the only possible law of attraction, *m* here denoting any commensurate number; but from this it does not follow that it fulfills all the conditions of the proposition for any *m*. In fact, one must have for values of α and β,

$$m\pi=\int_{\alpha}^{\beta}\frac{dz\sqrt{\frac{1}{\beta^{1/m^2-2}}-\frac{1}{\alpha^{1/m^2-2}}}}{\sqrt{\frac{\alpha^2}{\beta^{1/m^2-2}}-\frac{\beta^2}{\alpha^{1/m^2-2}}+\left(\beta^2-\alpha^2\right)\frac{1}{z^{1/m^2-2}}-z^2\left[\frac{1}{\beta^{1/m^2-2}}-\frac{1}{\alpha^{1/m^2-2}}\right]}},\quad(6)$$

[3]Let us assume initially $1/m^2-2$ negative; let us set $\alpha=0$, $\beta=1$, the equation becomes.

$$m\pi=\int_0^1\frac{dz}{\sqrt{\frac{1}{z^{1/m^2-2}}-z^2}}=\int_0^1\frac{z^{1/(2m^2)-1}dz}{\sqrt{1-z^{1/m^2}}},$$

and Eq. (6) yields

$$m\pi=m^2\pi,$$

$$m=1.$$

The corresponding law of attraction is

$$\varphi(r)=\frac{A}{r^2}.$$

If we assume $1/m^2-2$ positive, Eq. (6) for $\alpha=1$, $\beta=0$,

$$m\pi=\int_0^1\frac{dz}{\sqrt{1-z^2}}=\frac{\pi}{2}.$$

From this it follows that $m=1/2$, and the corresponding law of attraction is

$$\varphi(r)=Ar.$$

Only two laws fulfill the required conditions, that of nature, by which the closed orbit has only one symmetry axis through the centre of attraction, and the attraction proportional to the distance, according to which there are two [symmetry axes].

Our illustrious correspondent Mr. Chebychev[4], to whom I have communicated the precedent demonstration, judiciously made me remark that the theorem, useless nowadays for the so perfect theory of the planets, may be invoked in a useful way in order to extend to the double stars the laws of newtonian attraction.

III. CONCLUSIONS

More than a hundred years since its publication for the first time Bertrand's theorem still attracts the attention of the serious student of classical mechanics. Bertrand's proof is concise and elegant and can be assigned as a collateral reading or discussed in class. For most of the physics students around the world English is their second language (and of course, for many it is their first one!), therefore we hope that the present translation may be useful for those that decide to get acquainted with this important theorem.

Until the submission for publication of this translation, Bertrand's original paper could be found at the following address: http://gallica.bnf.fr/ark:/12148/bpt6k3034n.image.f849.langFR.

REFERENCES

[1] Bertrand, J., *Théorème relatif au mouvement d'un point attire vers un centre fixe*, C. R. Acad. Sci. Paris **77**, 849-853 (1873).

[2] Newton, I. *Philosophiae Naturalis Principia Mathematica*, (Royal Society, London, 1687). English translation by A. Motte revised by F. Cajori, (University of California Press, Berkeley CA, 1962).

[3] We have kept the original enumeration of the equations and for this reason there is no Eq. (5).

[4] In the original manuscript, Tchebychef.