Why a spaceship cannot reach the speed of light from the perspective of the spaceship's rest frame

Peter J. Riggs
Department of Quantum Science, Australian National University, Canberra ACT 0200, Australia.

E-mail: peter.riggs@anu.edu.au

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Abstract
A standard explanation in physics textbooks of why a spaceship, or any material body, cannot reach the speed of light in vacuum is that the increase with velocity of the spaceship's inertial mass is unlimited as light speed is approached, requiring infinite kinetic energy to reach the speed of light. However, this explanation is provided from the perspective of an inertial frame of reference moving with a constant (sub-light) speed with respect to the spaceship. What stops the spaceship from attaining light speed from the perspective of the spaceship itself, given that the spaceship's mass is constant in its rest frame? A suitable response is provided to this question.

Keywords: Speed of light, ultimate speed limit, rest mass, rest frame, relativistic effects.

Resumen
Una explicación habitual en los libros de texto de física de por qué una nave espacial, o cualquier cuerpo material, no pueden alcanzar la velocidad de la luz en el vacío es que el aumento de velocidad de la masa inercial de la nave espacial es ilimitado cuando se aproxima velocidad de la luz, lo que requiere energía cinética infinita para llegar a la velocidad de la luz. Sin embargo, esta explicación se ofrece desde la perspectiva de un sistema inercial de referencia que se mueve con una de velocidad constante (sub-luz) con respecto a la nave espacial. ¿Qué impide que la nave espacial alcance la velocidad de la luz desde la perspectiva de la propia nave espacial, dado que la masa de la nave espacial es constante en su marco de reposo? Se proporciona una respuesta adecuada a esta pregunta.

Palabras clave: Velocidad de la luz, límite de velocidad máxima, masa en reposo, sistema en reposo, los efectos relativistas.

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I. INTRODUCTION

Why cannot a spaceship ever reach the speed of light in vacuum? This question is regularly asked by students and interested members of the general public. Non-dynamical reasons commonly cited are the consistency of the Lorentz transformations and the maintenance of causality (*e.g*. [1, 2]). The standard dynamical explanation is that a spaceship's kinetic energy would have to become infinite. However, purely from the perspective of the spaceship (its rest frame), this dynamical explanation does not apply as its speed, and therefore kinetic energy, is zero. What then, relative to the spaceship's rest frame, stops it from reaching the speed of light? Students attempting to understand why the speed of light is the ultimate speed limit should be provided with a quantitative solution to this question, yet it does not appear to be in the textbooks. Further, the solution applicable in the spaceship's rest frame should be set out in terms of quantities that are (at least in principle) measurable in the rest frame.

II. SETTING THE PROBLEM

Assume that a spaceship is travelling through interstellar space in the positive x-direction of a Cartesian coordinate system of an inertial frame of reference S, whose origin may be taken to be at rest (e.g. the centre of the galaxy). Let the spaceship's velocity (denoted $\mathbf{u}$) in this frame have a magnitude in excess of 90% of the speed of light in vacuum (c, hereafter called light speed) so that relativistic effects are significant. Since the Special Theory of Relativity forbids any material body reaching light speed, all efforts by the crew of the spaceship to attain light speed must be unsuccessful. Historically, this circumstance has been explained in relativity monographs [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and also in general physics textbooks [13, 14, 15, 16, 17, 18, 19] by the unlimited increase in the inertial mass of the spaceship as light speed is approached. If the spaceship's inertial rest mass is m_o then its relativistic (or apparent) mass m, relative to frame S, is given by the wellknown relation: $m = m_o/(1 - u^2/c^2)^{1/2}$, with u = |**u**|. Then, as u $\rightarrow c$, $m \rightarrow \infty$ requiring the spaceship to have infinite kinetic energy (in frame S) in order to actually reach light speed, showing that this is a physical impossibility.

There has been a trend in recent years to reject the use of relativistic mass in favour of the use of relativistic momentum (*e.g*. [20, 21, 22, 23, 24, 25]). This view holds that the concept of relativistic mass is misleading and that a more suitable approach is to accept that an object's relativistic (three-) momentum $\boldsymbol{p}$, given by the relation: $\boldsymbol{p} = m_o\mathbf{u}/(1 - u^2/c^2)^{1/2}$, is not a *linear* function of speed, as with classical momentum [26] but tends to infinity as u $\rightarrow c$. Since both relativistic momentum and relativistic mass have to be infinite for an object to reach light speed (in frame S), either quantity can be used in a dynamical explanation. Note, however, that the solution with respect to the spaceship's rest frame (as presented below) does not depend on relativistic mass being accepted as a valid physical concept.

Returning to our imaginary spaceship, the crew could arrange for its engine (*e.g*. an ion ramjet drive [27]) to fire indefinitely in order to keep the spaceship accelerating. In the spaceship's rest frame, its mass is always the rest mass and its momentum is zero. Nevertheless, acceleration is measurable within the spaceship by means of an accelerometer. Ongoing acceleration in the direction of motion without an increase in the spaceship's mass (as in its rest frame) implies that *any* finite speed can eventually be reached. Why then, from the perspective of the spaceship, can it not attain light speed (or beyond)? The solution is not as obvious as the dynamical explanation (*e.g*. available in frame S) found in the textbooks.

Contrary to what was once believed, dealing with accelerating objects is not problematic within Special Relativity ([28, 29]), as stated by M.G. Bowler:

> *A frame of reference which is being accelerated by rockets firing is clearly not an inertial frame. This has given currency to the erroneous notion that special relativity is incapable of discussing the laws of physics experienced by accelerated observers. This idea is wholly incorrect: within the postulates of special relativity we have an unambiguous recipe for discussing such observers* [30].

The ingredients of this 'recipe' are laid out and employed in the treatment below.

Let another frame of reference (denoted S′) coincide with frame S at time t = 0 = t′ and let frame S′ be moving with speed v in the direction of the positive x-axis of frame S (*i.e*. the standard configuration for such frames [31]). This is depicted in Figure 1.

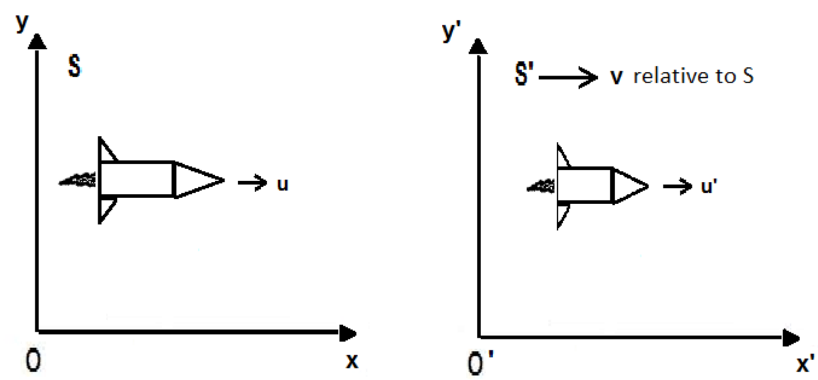

FIGURE 1. Representation of frames of reference S and S′.

Then the space and time coordinates of frame S are related to those in frame S′ by the familiar Lorentz transformations. We shall assume that none of the mass of the spaceship is ejected (which is possible using an ion ramjet) so that its rest mass remains constant. The speed of the spaceship u in frame S is related to its speed u′ in frame S′ by the Lorentz boost:

$$u = (u' + v) / (1 + vu'/c^2). \qquad (1)$$

The relation between the magnitudes of the acceleration in frame S (denoted a) and the acceleration in frame S′ (denoted a′) is found by differentiating Eq. (1):

$$a = (du/dt) = a' \{(1 - v^2/c^2)^{2/3} / (1 + vu'/c^2)^3\}$$
$$= a'/\{\gamma^3 (1 + vu'/c^2)^3\}, \qquad (2)$$

where $a' = (du'/dt')$ and $\gamma = \gamma(v) = 1/(1 - v^2/c^2)^{1/2}$.

The relativistic 'recipe' for dealing with acceleration involves using an instantaneous reference frame (IRF) and what is known as the Clock Postulate. An IRF is an inertial frame moving at the same speed as the spaceship at a given instant [32]. (Since the speed of the spaceship varies from instant to instant, we actually have a set (or family) of IRFs but this does not affect the mathematical treatment.) In its IRF, the speed of the spaceship is zero, *i.e*. u′ = 0 and v = u. Proper acceleration α is the acceleration measured in the IRF and is shown by the spaceship's accelerometer [33]. We shall assume, for simplicity, that the spaceship is subject to a constant force in the direction of motion (due to its engine's thrust) which provides *constant* proper acceleration. This is referred to as hyperbolic motion and its equations may be expressed in terms of α and either coordinate time t (in frame S) or proper time τ (which is shown by the spaceship's chronometer). In order for this to be the case, we must also accept the Clock Postulate which states that the rate of a clock only depends on the clock's instantaneous speed [34]. Although the equations of hyperbolic motion appear in several relativity texts (*e.g*. [35, 36, 37, 38, 39]), it will be helpful to quickly derive them. In the IRF, we find from Eq. (2):

$$(d\mathrm{u}/d\mathrm{t}) = \alpha/\gamma^3, \qquad (3)$$

as $\mathrm{u}' = 0$. Integrating Eq. (3) and taking u = 0 when t = 0 (*i.e*. the spaceship starts from rest) gives:

$$\mathrm{u(t)} = (d\mathrm{x}/d\mathrm{t}) = \alpha \mathrm{t}/\{1 + (\alpha \mathrm{t}/c)^2\}^{1/2}\,, \qquad (4)$$

since α is constant and where x is the spatial coordinate of the spaceship at time t in frame S. Integrating Eq. (4) and taking x = 0 when t = 0 gives:

$$\mathrm{x(t)} = (c^2/\alpha)\,\{[1 + (\alpha \mathrm{t}/c)^2]^{1/2} - 1\}. \qquad (5)$$

The relation of proper time τ to coordinate time t may be found from the metric of Minkowski spacetime which provides the differential of proper time:

$$d\tau = ds/c = \{d\mathrm{t}^2 - (d\mathrm{x}^2 + d\mathrm{y}^2 + d\mathrm{z}^2)/c^2\}^{1/2} = (1 - \mathrm{u}^2/c^2)^{1/2}\, d\mathrm{t}.$$

On substitution from Eq. (4) and integrating, we get:

$$\tau = \int\{1 - [\alpha^2\mathrm{t}^2/c^2\,(1 + \alpha^2\mathrm{t}^2/c^2)]\}^{1/2}\, d\mathrm{t},$$
$$= (c/\alpha)\,\mathrm{arcsinh}\,(\alpha \mathrm{t}/c), \qquad (6)$$

where the constant of integration is zero. Using Eq. (6), Eq. (4) may be written in terms of α and τ, as follows:

$$\mathrm{u}(\tau) = c\,\tanh\,(\alpha\tau/c)\,. \qquad (7)$$

The value of u being less than c is apparent from Eqs. (4) and (7) for, given that both c and α are constant, it can be seen that the limit of u, as t (or τ) tends to infinity, is c. In other words, it would take an infinite amount of time for the spaceship to reach light speed so that for any finite time, $\mathrm{u} < c$. Also, since α is constant, Eq. (3) shows that the acceleration in frame S *decreases* as the spacecraft's speed increases. This was to be expected as the spacecraft's relativistic mass increases markedly in frame S as light speed is closely approached.

Although Eqs. (3)-(7) accurately *describe* hyperbolic motion, they do not provide a clear *physical* explanation for why light speed is not attainable. Also, since the spaceship's mass is constant and its momentum is zero in its rest frame, the quantities of mass and momentum cannot assist in formulating a solution to the question posed that is valid in that frame. An answer is presented below which, although previously flagged [40, 41], is not widely acknowledged and has not been published as a rigorous solution.

III. EXPLANATION IN THE SPACESHIP'S REST FRAME

Consistent with the Principle of Relativity, there is no experiment that can be conducted entirely within a closed (non-accelerating) spaceship which will be able to *measure* its speed with respect to an external inertial frame. However, the spaceship's crew can calculate the instantaneous speed u whilst the spaceship accelerates by noting the values of α and τ and substituting these into Eq. (7) [42]. In frame S, let the total interstellar distance travelled by the spaceship be D and the complete travel duration be T, *i.e*. distance D has been traversed in time T. Whilst accelerating, let the spaceship's IRF be denoted S″. In this frame, the distance travelled to the destination and its travel duration shall be denoted as $\mathcal{D}$ and $\mathcal{T}$ respectively. Given that the origins of the frames S and S′ (and therefore frame S″) coincide at time t = 0 = t′, we have from Eq. (6):

$$\mathcal{T} = (c/\alpha)\,\mathrm{arcsinh}\,(\alpha \mathrm{T}/c)\,, \qquad (8)$$

and from Eq. (5):

$$\mathrm{D} = \mathrm{x(T)} = (c^2/\alpha)\,\{[1 + (\alpha \mathrm{T}/c)^2]^{1/2} - 1\}. \qquad (9)$$

If the spaceship's speed was always constant then spatial distances as measured in frames S and S″ would be related by the usual relativistic length contraction equation. However, in the case of non-constant speed, it is differentials of these spatial distances (dx and dx″ respectively) that are related by the Lorentz factor γ:
$d\mathrm{x} = \gamma(\mathrm{u})\, d\mathrm{x}''$, where $\gamma(\mathrm{u}) = 1/(1 - \mathrm{u}^2/c^2)^{1/2}$ and the double prime refers to frame S″. Then the total distance traversed by the accelerating spaceship as measured in frame S″ is:

$$\mathcal{D} = \int_0^{\mathrm{D}} (1 - \mathrm{u}^2/c^2)^{1/2}\, d\mathrm{x} = \int_0^{\mathrm{T}} \mathrm{u}\,(1 - \mathrm{u}^2/c^2)^{1/2}\, d\mathrm{t},$$

since $d\mathrm{x} = \mathrm{u}\, d\mathrm{t}$. Therefore,

$$\mathcal{D} = \int_0^{\mathrm{T}} \{\alpha c\mathrm{t}/(c^2 + \alpha^2\mathrm{t}^2)^{1/2}\}\,\{c/(c^2 + \alpha^2\mathrm{t}^2)^{1/2}\}\, d\mathrm{t},$$
$$= \int_0^{\mathrm{T}} \alpha c^2\mathrm{t}/(c^2 + \alpha^2\mathrm{t}^2)\, d\mathrm{t},$$
$$= (c^2/2\alpha)\,\big[\log_e\{1 + (\alpha^2\mathrm{t}^2/c^2)\}\big]_0^{\mathrm{T}}\,,$$
$$= (c^2/2\alpha)\,\log_e\{1 + (\alpha^2\mathrm{T}^2/c^2)\}.$$

From Eq. (9), we have:

$$T^2 = (c^2/\alpha^2)\{[(\alpha D/c^2) + 1]^2 - 1\},$$

and substituting for T^2, we find the relation between $\mathcal{D}$ and D:

$$\mathcal{D} = (c^2/\alpha)\log_e[(\alpha D/c^2) + 1]. \quad (10)$$

It can be seen from Eqs. (8) and (10) that $\mathcal{T} < T$ and $\mathcal{D} < D$, as would be expected due to time dilation and length contraction effects. How much smaller $\mathcal{D}$ and $\mathcal{T}$ are will depend on how close to light speed is achieved. Numerical calculations for astronomical distances may be found in relevant textbooks and on the internet (*e.g.* [43, 44]).

The answer to the question posed is as follows. Let the speed u at time t = T be denoted U. Then, from Eq. (4), $U = u(T) = \alpha T/\{1 + (\alpha T/c)^2\}^{1/2}$. We can now express U in terms of $\mathcal{D}$ and $\mathcal{T}$, as (from Eqs (8) and (10)), $\alpha T = c \sinh(\alpha\mathcal{T}/c)$ and $\{1 + (\alpha T/c)^2\}^{1/2} = \exp[\alpha\mathcal{D}/c^2]$. These give:

$$U = c \sinh(\alpha\mathcal{T}/c) / \exp[\alpha\mathcal{D}/c^2], \quad (11)$$

which indicates that $U < c$ if both $\mathcal{T}$ and $\mathcal{D}$ are finite. In order to see that this is indeed the case, Eq. (11) may be rewritten only in terms of $\mathcal{D}$:

$$U = c\,\{\exp[2\alpha\mathcal{D}/c^2] - 1\}^{1/2} / \exp[\alpha\mathcal{D}/c^2],$$

from which it is obvious that U must be strictly less than light speed for finite values of $\mathcal{D}$. However, the crucial result follows from Eq. (11), which is that *both the spatial and temporal intervals* in the spaceship's frame of reference vary such that its speed will always be less than light speed even though the spaceship can accelerate indefinitely. This result depends on time dilation *and* length contraction which are consequences of the structure of Minkowski spacetime [45, 46, 47]. Therefore, we can account for light speed being the 'ultimate speed' from the perspective of the spaceship purely in terms of the consequences of spacetime structure, without the need to refer to mass or momentum.

IV. CONCLUSIONS

A quantitative explanation has been offered for why a spaceship (or other body) cannot reach the speed of light from the perspective of the spaceship's rest frame. The solution results as a consequence of the structure of Minkowski spacetime and has been given in terms of quantities that are measurable in the rest frame. This solution provides a physical account that will assist students in gaining an understanding of why light speed is not attainable regardless of an observer's frame of reference.

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