Experiments with Polygonal and Polyhedral Resistive Structures

R. Beltrán, F. Gómez, R. Franco, J-Alexis Rodríguez and F. Fajardo
Departamento de Física, Universidad Nacional de Colombia, Bogotá-Colombia.

E-mail: fefajardot@unal.edu.co

(Recibido el 18 de Septiembre de 2013; aceptado el 27 de Diciembre de 2013)

Abstract

In this work the equivalent resistors of eight regular polygonal and five polyhedral resistive structures are studied experimentally. The structures were constructed using commercial precision resistors of 1.209 kΩ ±0.5 % and the resistances were measured using multimeters available in any physics laboratory. The resistance dependence of the polygonal structures with relation to the vertex number and its convergence was determined. For polyhedral structures the resistances were measured across nodes. We found that the experimental equivalent resistance for all the structures shows good concordance with the theoretical predictions. This experiment offers the opportunity to shows students the use of symmetry in physics.

Keywords: Resistive Structures, Equivalent Resistance, Polyhedral Structures.

Resumen

En este trabajo la resistencia equivalente de ocho polígonos regulares y de ocho estructuras poliédricas resistivas son estudiadas experimentalmente. Las estructuras fueron construidas utilizando resistores comerciales de precisión de 1.209 kΩ ±0.5% y las resistencias se midieron usando multimetros disponibles en cualquier laboratorio de física. La dependencia de la resistencia de las estructuras poligonales con relación al número de vértices y su convergencia fue determinada. Para las estructuras poliédricas fue medida su resistencia entre los nodos. Encontramos que la medida de la resistencia equivalente para todas las estructuras muestra una buena concordancia con las predicciones teóricas. Este experimento ofrece la oportunidad de mostrar a los estudiantes el uso de la simetría en física.

Palabras Clave: Estructuras Resistivas, Resistencia Equivalente, Estructuras Poliédricas.

PACS: 01.50.Pa, 01.55.+b **ISSN 1870-9095**

I. INTRODUCTION

In introductory physics courses resistors is used in a wide variety of electricity experiments, such as in the characterization of resistors in series and parallel connections or to investigate the relationship between current and voltage. For more advanced students is possible to utilized resistors in more interesting and formative experiments on electricity. In this way the use of resistor circuits in order to determine the resistance between two nodes in different geometric configurations or structures is possible, among which we can mention: the ladder network, square or hexagonal infinite networks, polygons and polyhedral resistive structures [1, 2, 3, 4, 5]. A typical exercise is to find the resistance between vertices at the ends of a long diagonal when the edges of a cube are replaced by equal resistors. In a cube for each of the end points the three adjacent vertices are at the same potential, because of the cube symmetry under a 120^{o} rotation around the long diagonal. Therefore the cube network is thus equivalent to one in which three resistors in parallel are in series with six resistors in parallel and with three resistors in parallel [6]. In the particular case of the 1 ohm cube network a total resistance of 1/3 + 1/6 + 1/3 = 5/6 ohms is obtained. The didactic interest of the exercise is related to the use of the structure symmetries and then the students have to recognize which vertices have the same potential. This exercise can be modified to consider a tetrahedron in 3-dimensions but it could be reduced to 2-dimensions as three sided polygon with equal resistors R in each side and also each vertex connected to the center with the same resistance R. For that structure the equivalent resistance obtained is $R/2$. The experimental setup for this exercise can be performed with equipment available in every physics laboratory and the resistors are of easy consecution.

Next we do a quick revision of some results found in the literature to calculate the equivalent resistance of polygonal and polyhedral structures on which is based this experiment. Afterwards we present the setup and the experimental results that were found for the equivalent resistance of polygonal and polyhedral resistive structures.

II. THEORY

There are many theoretical approaches for the polygonal and polyhedral resistive structures in the literature [1, 2, 3, 4, 5, 6, 7, 8]. Polygonal resistive structures are related to the Fibonacci sequence and it has been shown that for a large number of sides, *N*, the whole network of 2*N* resistors has a vertex-to center resistance of $R/\sqrt{5}$ and it is a consequence that for large *N*, the ratio of successive Fibonacci terms (F_N) is the golden ratio $\phi = (1+\sqrt{5})/2$ [2, 7]. For a polygonal network of *N* sides linked with equal R_o resistances with *N* an odd number after a recursive procedure the equivalent resistance can be obtained [2] and it is

$$R_N = R_o \frac{F}{F_{N-1}+F_{N+1}} \tag{1}$$

Using $F_N = (\phi^N - (-\phi)^{-N})/\sqrt{5}$ and the property $\phi - 1 = \phi^{-1}$, it is possible to reduce the above expression to $1/(2\phi - 1) = 1/\sqrt{5}$. A similar result can be obtained for *N* an even number of sides,

$$R_N = R_o \frac{F_{N-1}+F_{N+1}}{2(F_{N-1}+F_{N+1})+F_{N-2}+F_{N-4}} \tag{2}$$

and also it can be reduced to $(1+2\phi)/(3+4\phi) = 1/\sqrt{5}$. Therefore for a large number of sides *N* the effective resistance converges to $R_o/\sqrt{5}$.

III. EXPERIMENT

It is possible to assemble simple circuits in the laboratory to test the theoretical predictions for polygonal structures given by equations (1) and (2). A simple set up of eight polygons between 3 and 10 sides were building up. In figure 1 are shown four of the polygon resitors structures used in the experiment. The resistive structures were made using commercial precision resistors of R_o = 1.209 kΩ ± 0.5% and the resistances were measured using an ohmmeter with an accuracy of ± 0.0005 $k\Omega$. We recommended to test that each junction in the structures was perfectly welded in order to ensure good results in the measurements of the equivalent resistances

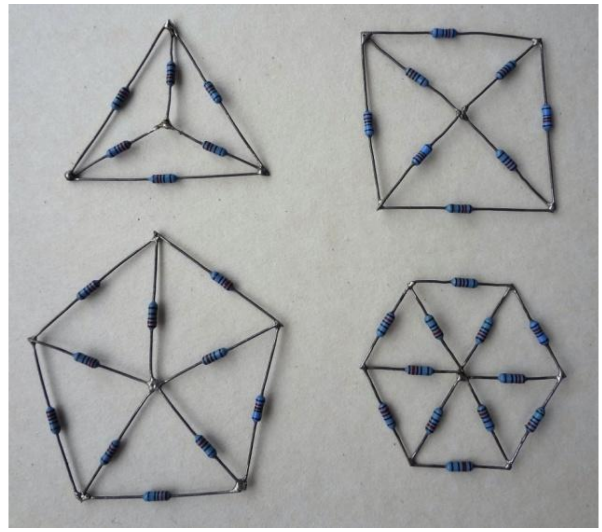

FIGURA 1. Some examples of the set of the resistive polygons considered in the experiment. The resistance of each side is R_0 = 1.209 kΩ ± 0.5%.

IV. RESULTS

Measurements of the polygon resistors structures were done for all vertices and the results are summarized in table I, where are compared the experimental averages equivalent resistance with the theoretical value. Note that the percentage of error in all cases is less than 0.2 %. On the other hand, using data from table I, in figure 2 is represented $(R_N/R_o)\sqrt{5}$ as a function of the number of sides *N*. The squares represent the experimental averaged value for R_N^{Exp} and triangles are the theoretical resistances R_N^{Theor}. To notice the asymptotic convergence when the number of sides *N* is increased and this fact is in agreement with the theoretical expectation given by equations (1) and (2).

TABLE I. Results for the polygon resistor structures, where: N is the number of sides, R_N^{Exp} is the experimental averages equivalent resistance, R_N^{Theor} is the theoretical value given by equations (1),(2), and E is the percentage error.

N	R_N^{Exp} (kΩ)	R_N^{Theor} (kΩ)	E (%)
3	0.6038	0.6045	0.12
4	0.5640	0.5642	0.04
5	0.5490	0.5495	0.10
6	0.5435	0.5441	0.10
7	0.5417	0.5420	0.05
8	0.5409	0.5418	0.05
9	0.5410	0.5409	0.02
10	0.5407	0.5408	0.01

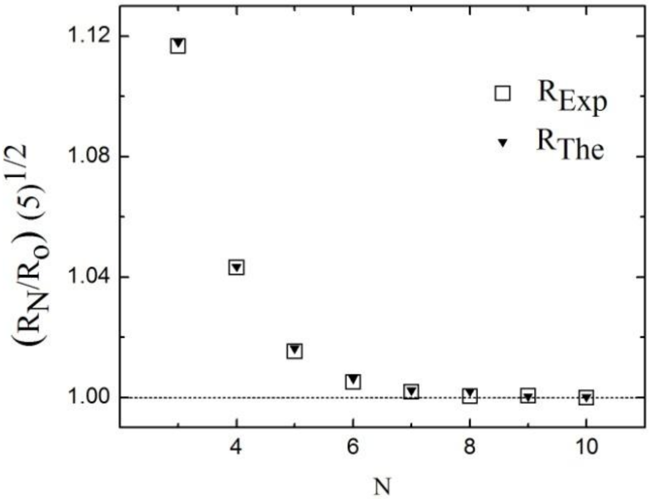

FIGURE 2. $\left(R_N^{Exp}/R_o\right)\sqrt{5}$ and $\left(R_N^{Theo}/R_o\right)\sqrt{5}$ as a function of N, with R_N the equivalent resistance of the N side's polygon [experimental (□) and theoretically (▼) values].

Other interesting resistive networks are the three dimensional structures known as polyhedral structures [4], they are structures with a high degree of symmetry which can be used in order to determine the equivalent resistor between any two vertices of the regular polyhedral structure under consideration. The theoretical approach is based on the identification of the planes through equipotential vertices and again with an iterative procedure, the equivalent resistor is determined by [4],

$$R_i = \frac{2R_o}{H}\sum_{j=1}^{i}\frac{1}{n_j}\left(H - \sum_{k=0}^{j-1} q_k\right) \tag{3}$$

where H is the number of vertices of the regular polyhedral structures, i is the smallest number of resistors that has to be transverse when going from the vertex of the incoming current to the vertex of the outgoing current, n_j is the number of resistors in parallel between the equipotential planes j-1 and j, q_k is the number of vertices in the equipotential plane k, and finally R_o is the resistance along the edges of the polyhedral structure. The characteristic numbers of polyhedron and polyhedral resistive structures (see reference [4]) are shown in table II and in figure 3 are shown the polyhedral structures building in this work. The experimental test of the equivalent resistance of polyhedral structures given by equation (3) is setting up using commercial precision resistors R_o = 1.209 $k\Omega$ ± 0.5% and building up the different polyhedron. The results are also shown in table II where are presented the experimental equivalent resistances and the calculated one using equation (3), also is shown the percentage error E; notice that the agreement theory-experiment is notorious.

On the other hand, in figure 4 the equivalent resistance *R(i)* of different polyhedral structures is plotted as a function of the number *i* of the first neighbors. We can see that the equivalent resistance of the polyhedral structures is increasing when the number *i* is increased as it is expected. It is interesting to notice that the curves are from the lower one to the upper one as the icosahedron (*H =12*), then the octahedron (*H =6*), just a dot for the tetrahedron (*H =4*), then the cube (*H =8*) and finally the dodecahedron (*H =20*).

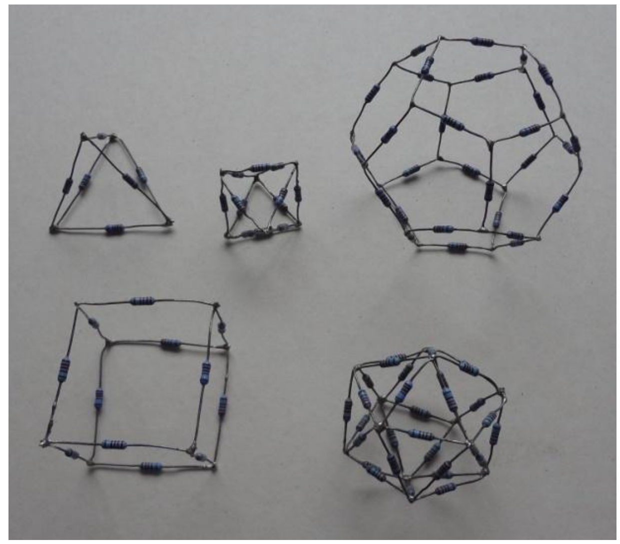

FIGURE 3. Polyhedral resistive structures: Tetrahedron, Octahedron, Icosahedron, Cube, Dodecahedron. The resistance of each side is R_0 = 1.209 kΩ ± 0.5%.

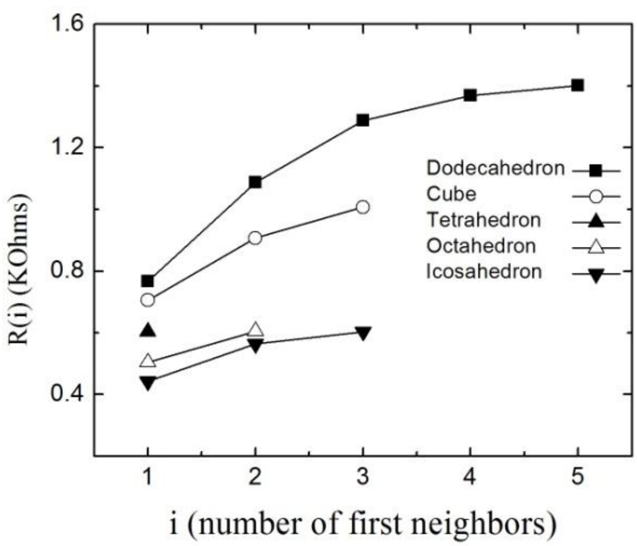

FIGURE 4. Equivalent resistance R(i) of different polyhedral structures as a function of the number of first neighbors i for each polyhedra. Lines have been drawn as a guide only.

That fact is related to how many equipotential planes has the resistive structure and how many resistors in parallel are between these planes. In general the behavior of the equivalent resistance *R(i)* is related to the symmetries of the polyhedral structure which can be seen in the different numbers n_j. The numbers n_j are the numbers of resistors in parallel between the equipotential planes j-1 and j, as can be seen in table II.

V. CONCLUSIONS

Simple experiments related with polygonal and polyhedral resistive structures can be set up in order to use the concept of symmetry of a particular arrangement of resistors to obtain the equivalent resistance of the structure. Also, they can be useful to complement the standard Ohm's Law experiments on introductory physics courses. The experiments presented have been set up using low cost materials and the standard equipment available in any physics lab. We found that measurements of the equivalent resistance for both polygonal and polyhedral resistive structures are in excellent agreement with the predicted values.

TABLE II. Some characteristic numbers associated to different polyhedral resistive structures: tetrahedron, octahedron, icosahedron, cube and dodecahedron. Where H is the number of vertices in the polyhedra, i is the lower number of resistors between the inside vertex and the outside one for each polyhedra, n_j is the number of resistors between the equipotential planes j-1 and j, and q_k is the number of vertices in the equipotential plane k. The experimental results of the equivalent resistor for the different polyhedral structures are also shown: $R^{Th}(i)$ is the theoretical value, $R^{Exp}(i)$ is the experimental averaged resistance, and E is the percentage error. The resistance $R_0 = 1.209$ kΩ ± 0.5%.

Polyhedra	H	i	$(n_1,\ldots,n_j)$ $(q_o,\ldots,q_k)$	$R^{Th}(i)$ (kΩ)	$R^{Exp}(i)$ (kΩ)	E (%)
Tetrahedron	4	1	(3) (1,3)	$\frac{1}{2}Ro = 0.6045$	0.6036	0.15
Octahedron	6	1	(4,4) (1,4,1)	$\frac{5}{12}Ro = 0.5038$	0.5038	0.00
		2		$\frac{1}{2}Ro = 0.6045$	0.6050	0.08
Icosahedron	12	1	(5,10,5) (1,5,5,1)	$\frac{11}{30}Ro = 0.4433$	0.4425	0.18
		2		$\frac{7}{15}Ro = 0.5642$	0.5635	0.12
		3		$\frac{1}{2}Ro = 0.6045$	0.6040	0.08
Cube	8	1	(3,6,3) (1,3,3,1)	$\frac{7}{12}Ro = 0.7053$	0.7057	0.06
		2		$\frac{3}{4}Ro = 0.9068$	0.9073	0.05
		3		$\frac{5}{6}Ro = 1.0075$	1.0080	0.05
Dodecahedron	20	1	(3,6,6,6,3) (1,3,6,6,3,1)	$\frac{19}{30}Ro = 0.7657$	0.7660	0.04
		2		$\frac{9}{10}Ro = 1.0881$	1.0870	0.10
		3		$\frac{16}{15}Ro = 1.2896$	1.2880	0.12
		4		$\frac{17}{15}Ro = 1.3702$	1.3693	0.06
		5		$\frac{7}{6}Ro = 1.4105$	1.4090	0.11

REFERENCES

[1] Atkinson, D. and Van Steenwijk, F. J., *Infinite resistive lattices*, Am. J. Phys. **67**, 486-492 (1999).
[2] March, R. H., *Polygons of resistors and convergent series*, Am. J. Phys. **61**, 900-901 (1993).
[3] Sidhu, S. S., *Polygons of unequal resistors*, Am. J. Phys. **62**, 815-816 (1994).
[4] VanSteenwijk, F. J., *Equivalent resistors of polyhedral resistive structures*, Am. J. Phys. **66**, 90-91 (1998).
[5] Denardo, B., Earwood, J. and Sazonova, V., *Experiments with electrical resistive networks*, Am. J. Phys. **67**, 981-986 (1999).
[6] Pippenger, N., *The hypercube of resistors, asymptotic expansion and preferential arrangements*, Mathematics Magazine **83**, 331-346 (2010).
[7] Srinivasan, T. P., *Fibonacci sequence, golden ratio and a network of resistors*, Am. J. Phys. **60**, 461-462 (1992).
[8] Higbie, J., *Proof of the R-2R ladder network*, The Physics Teacher **35**, 464-465 (1997).