Finding the maximum and minimum magnitude responses (gains) of third-order filters without calculus

EDVCATIO PHYSICORVM
QVO NON ASCENDAM?

Kenneth V. Cartwright[1], **Patrick Russell**[1] and **Edit J. Kaminsky**[2]
[1]*School of Mathematics, Physics, and Technology, College of The Bahamas, P.O. Box N4912, Nassau, Bahamas.*
[2]*Department of Electrical Engineering, EN 846 Lakefront Campus, University of New Orleans, New Orleans, LA 70148, USA.*

E-mail: kvcartwright@yahoo.com

(Received 1 July 2013, accepted 3 December 2013)

Abstract

The maximum and minimum gains (with respect to frequency) of third-order low-pass and high-pass filters are derived without using calculus. Our method uses the little known fact that extrema of cubic functions can easily be found by purely algebraic means. PSpice simulations are provided that verify the theoretical results.

Keywords: Filters, Maximum and minimum without calculus, PSpice simulation.

Resumen

Derivamos las ganancias máxima y mínima (con respecto a la frecuencia) de filtros de tercer orden de paso bajo y de paso alto sin usar cálculo. Nuestro método utiliza el hecho poco conocido que los extremos de funciones cúbicas pueden encontrarse fácilmente con métodos puramente algebraicos. Verificamos los resultados teoréticos con simulaciones en PSpice.

Palabras Clave: Filtros, Máximo y mínimo sin cálculo, simulaciones en PSpice.

PACS: 84.30.Vn, 02.90.+p, 01.40.Ha and 01.40.Fk. **ISSN 1870-9095**

I. INTRODUCTION

Third-order filters are common in electronics, and for a subgroup of third-order filters, two important features are the maximum and minimum of the magnitude responses (gains) attainable by the filters, whether they are low-pass or high-pass filters. Conventionally, the mathematical expressions for these extrema values are found with calculus, or indeed simply stated without derivation. However, this puts the curious student who has not yet had the chance to study calculus at a disadvantage. Presently, he or she has no choice but to accept the equations for the maximum and minimum gains without any understanding as to their origins. Fortunately, as we show in this paper, it is straightforward to derive these maximum and minimum gains without calculus. To do this, we use the little known fact that extrema of cubic functions can be found through algebraic means alone [1, 2, 3, 4]. Furthermore, PSpice simulations will be used to verify the theoretical equations. (PSpice is a popular electrical and electronic circuits simulation software package that is widely used by electrical engineers and some physicists. The latest demo version can be freely obtained from [5]).

II. GAIN OF THIRD-ORDER FILTERS

In this section, the gain of third-order filters is given.

A. Low-Pass Filter

The transfer function of a general third-order low-pass filter is given by

$$T(s)=\frac{d}{s^3+as^2+bs+c}, \tag{1}$$

where a,b,c,d are constants and $s=i\omega$, with $i=\sqrt{-1}$, and ω is the angular frequency of the applied sine-wave. For example, the transfer function of a third-order 1-dB Chebyshev low-pass filter with cut-off frequency of 1 rad/s is given by (pg.13 of [6])

$$T(s)=\frac{0.4913}{s^3+0.9883s^2+1.2384s+0.4913}. \tag{2}$$

The gain of the third-order low-pass filter is simply the magnitude of Eq. (1), *i.e.*,

$$|T(\omega)| = \frac{|d|}{\sqrt{\left(-a\omega^2+c\right)^2+\omega^2\left(-\omega^2+b\right)^2}}. \quad (3)$$

A sketch of Eq. (3) is shown in Fig. 1, for the subgroup of third-order low-pass filters in which we are interested, where the following are clearly identified:

(i) maximum gain, $|T(\omega)|_{max}$,
(ii) the frequency at which the maximum gain occurs, ω_{max},
(iii) minimum gain, $|T(\omega)|_{min}$, and
(iv) the frequency at which the minimum gain occurs, ω_{min}.

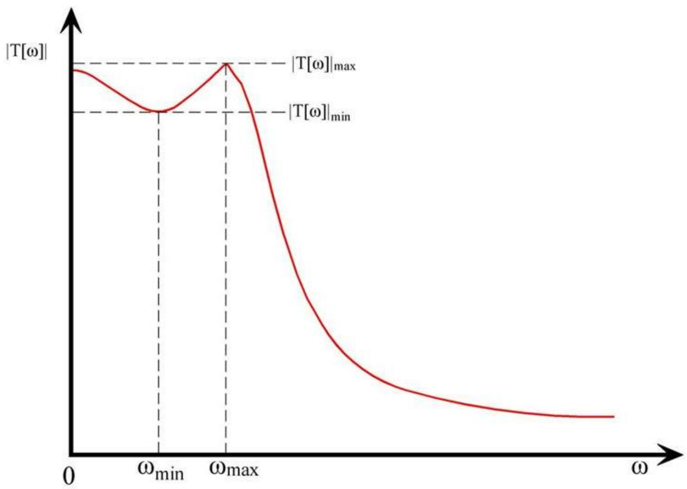

FIGURE 1. Magnitude response (gain) $|T(\omega)|$ of a third-order low-pass filter with ripple in the pass-band.

B. High-Pass Filter

The transfer function of a general third-order high-pass filter is given by

$$T(s) = \frac{ds^3}{s^3+as^2+bs+c}, \quad (4)$$

For example, the transfer function of a third-order 1-dB Chebyshev high-pass filter with cut-off frequency of 1 rad/s can be derived from Eq. (2) by using the low-pass to high-pass transformation method, *i.e.*, replacing s with $1/s$ (see Example 8.5.1 of [7]) to get

$$\begin{aligned} T(s) &= \frac{0.4913}{\left(\frac{1}{s}\right)^3+0.9883\left(\frac{1}{s}\right)^2+1.2384\left(\frac{1}{s}\right)+0.4913} \\ &= \frac{s^3}{s^3+2.5206s^2+2.0117s+2.0354}. \end{aligned} \quad (5)$$

The gain of the third-order high-pass filter is the magnitude of Eq. (4), *i.e.*,

$$|T(\omega)| = \frac{|d|\omega^3}{\sqrt{\left(-a\omega^2+c\right)^2+\omega^2\left(-\omega^2+b\right)^2}}. \quad (6)$$

A sketch of Eq. (6) is shown in Fig. 2, for the subgroup of third-order high-pass filters in which we are interested, where, again, (i) through (iv) are clearly identified:

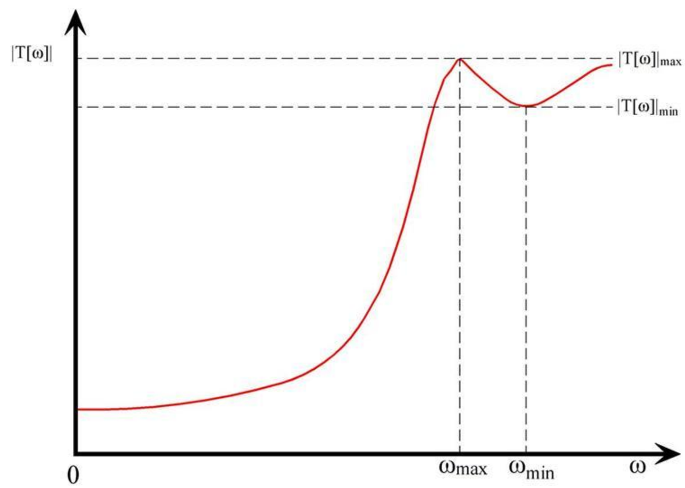

FIGURE 2. Magnitude response (gain) $|T(\omega)|$ of a third-order high-pass filter with ripple in the pass-band.

III. DETERMINING THE MAXIMUM AND MINIMUM GAINS OF THIRD-ORDER FILTERS WITHOUT USING CALCULUS

Now that the gains of third-order low-pass and high-pass filters have been stated, we can show how the maximum and minimum gains can be determined without using calculus. We will use the following fact that is traditionally established with calculus, but which is easily verified by purely algebraic means [1, 2, 3, 4], as shown in the Appendix.

Fact: Suppose $f(x) = Ax^3+Bx^2+Cx+D$, then the minimum or maximum value of $f(x)$ occurs when x is a root of $3Ax^2+2Bx+C$, *i.e.*, $3Ax^2+2Bx+C=0$.

A. Low-Pass Filter

From Eq. (3), the square of the gain of the third-order low-pass filter is

$$\begin{aligned}|T(\omega)|^2 &= \frac{d^2}{\left(-a\omega^2+c\right)^2+\omega^2\left(-\omega^2+b\right)^2} \\ &= \frac{d^2}{\omega^6+\left(a^2-2b\right)\omega^4+\left(b^2-2ac\right)\omega^2+c^2} \\ &= \frac{d^2}{x^3+Bx^2+Cx+D},\end{aligned} \tag{7}$$

where $x=\omega^2$, $B=a^2-2b, C=b^2-2ac,$ and $D=c^2$.

Clearly, the minimum/maximum value of Eq. (3) or Eq. (7) occurs when the denominator of Eq. (7) is a maximum/minimum value, *i.e.*, at the solutions of $x^2+\frac{2B}{3}x+\frac{C}{3}=0.$ Hence,

$$\omega_{\min or \max}=\sqrt{\frac{-B\pm\sqrt{B^2-3C}}{3}}. \tag{8}$$

Note that we must have $B^2>3C$ in order for there to be a maximum and minimum gain.

Substituting Eq. (8) into Eq. (7) gives the square of the maximum or minimum gain of the third–order low-pass filter.

B. High-Pass Filter

From Eq. (6), the square of the gain of the third-order high-pass filter is

$$\begin{aligned}|T(\omega)|^2 &= \frac{d^2\omega^6}{\left(-a\omega^2+c\right)^2+\omega^2\left(-\omega^2+b\right)^2} \\ &= \frac{d^2\omega^6}{\omega^6+\left(a^2-2b\right)\omega^4+\left(b^2-2ac\right)\omega^2+c^2} \\ &= \frac{d^2}{1+\left(a^2-2b\right)\omega^{-2}+\left(b^2-2ac\right)\omega^{-4}+c^2\omega^{-6}}.\end{aligned} \tag{9}$$

Letting $x=\omega^{-2}$, $B=\left(b^2-2ac\right)/c^2$, $C=\left(a^2-2b\right)/c^2$, and $D=1/c^2$, we get

$$|T(x)|^2=\frac{(d/c)^2}{x^3+Bx^2+Cx+D}. \tag{10}$$

Clearly, the minimum/maximum value of Eq. (6) or Eq. (9) occurs when the denominator of Eq. (10) is a maximum/minimum value, *i.e.*, at the solutions of $x^2+\frac{2B}{3}x+\frac{C}{3}=0.$ Hence,

$$\omega_{\min or \max}=\frac{1}{\sqrt{\frac{-B\pm\sqrt{B^2-3C}}{3}}}. \tag{11}$$

Note again that we must have $B^2>3C$ in order for there to be a maximum and minimum gain.

Substituting Eq. (11) into Eq. (9) gives the square of the maximum or minimum gain of the third–order high-pass filter.

IV. PSPICE SIMULATIONS

In this section, we will use PSpice simulations to verify the equations for the extrema of the gains and the frequencies at which they occur for the third-order low-pass and high-pass filters.

A. Low-Pass Filter

For the low-pass filter example used in Eq. (2), $a=0.9883$, $b=1.2384$, $c=0.4913$, and $d=c$. Hence, $B=-1.500$ and $C=0.5625$. So, from Eq. (8), $\omega_{\min or \max}=0.86605$ rad/s (137.84 mHz) or 0.50000 rad/s (79.577 mHz). Furthermore, substituting these values into Eq. (7) and taking the square root gives $|T(\omega)|_{\max or \min}=1.0000$ (0 dB) or 0.89124 (-1.0000 dB).

To check these theoretical values, a PSpice simulation of the third-order Chebyshev 1.0 dB low-pass filter was done using the diagram shown in Fig. 3.

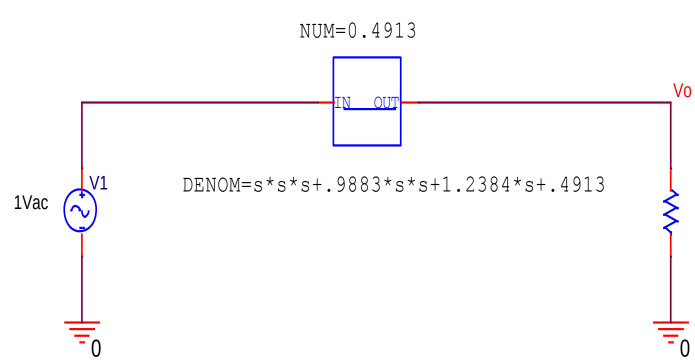

FIGURE 3. PSpice implementation of a third-order Chebyshev 1.0 dB low-pass filter with cut-off frequency of 1 rad/s.

Upon completion of the simulation, the PSpice magnitude response was plotted in Fig. 4, using 1000 points per decade. From this figure, it is clear that (i) Eq. (2) is in fact the transfer function of a third-order Chebyshev 1.0 dB low-pass filter; otherwise, the magnitude response would be different from that observed, and (ii) the theoretical values are very close to the simulated values given in the graph, as summarized in Table I.

TABLE I. Comparison of theoretical and PSpice simulated values.

Parameter	Theoretical Value	Simulated Value
f_{cutoff}	159.155 mHz	159.218 mHz
$f_{\min}$	79.577 mHz	79.686 mHz
$f_{\max}$	137.84 mHz	137.53 mHz
$\lvert T(\omega)\rvert_{\min}$	-1.0000 dB	-1.0000 dB
$\lvert T(\omega)\rvert_{\max}$	0 dB	4.3350x10^{-4} dB

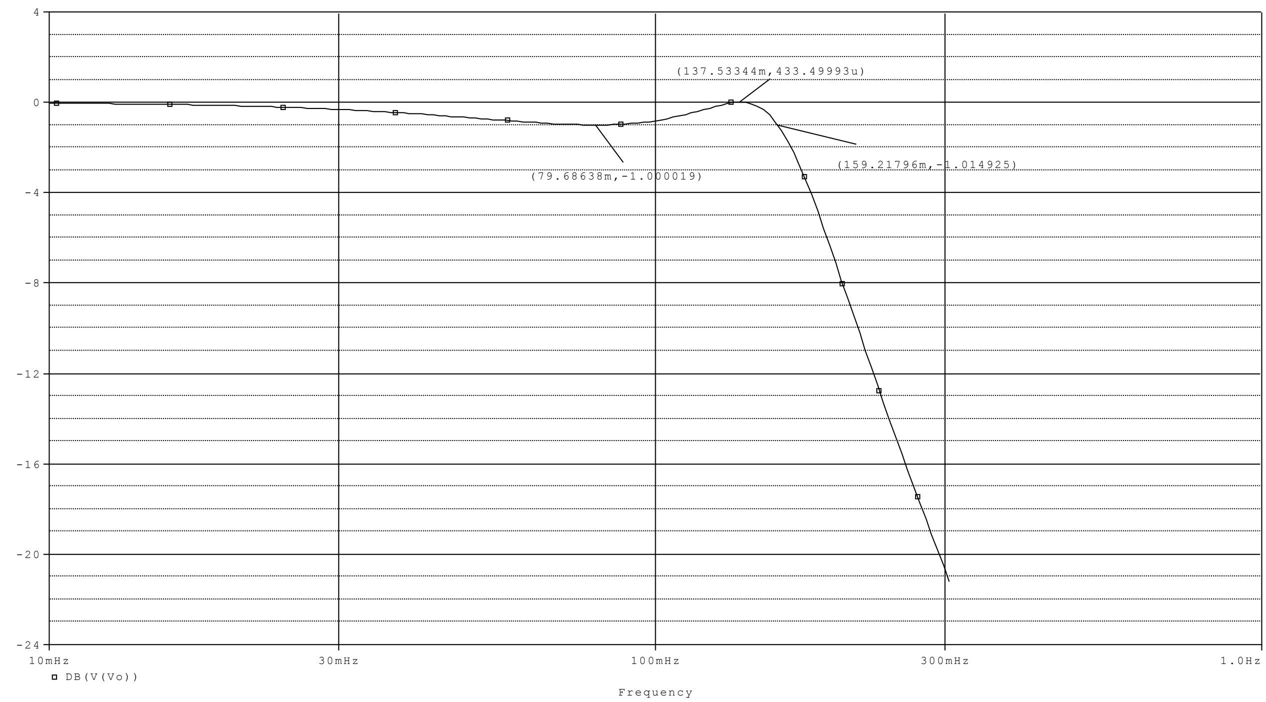

FIGURE 4. PSpice magnitude response (gain in dB) of a third-order Chebyshev 1.0 dB low-pass filter, as simulated with Fig. 3.

B. High-Pass Filter

For the high-pass filter example used in Eq. (5), $a = 2.5206$, $b = 2.0117$, $c = 2.0354$ and $d = 1$. Hence, $B = -1.499$ and $C = 0.5624$. So, from Eq. (11), $\omega_{\min\,or\,\max} = 1.1547$ rad/s (183.78 mHz) or 2.0001 rad/s (318.33 mHz). Furthermore, substituting these values into Eq. (9) and taking the square root gives $\lvert T(\omega)\rvert_{\max\,or\,\min} = 1.0000$ (0 dB) or 0.89108 (-1.0017 dB).

To check these theoretical values, a PSpice simulation of the third-order Chebyshev 1.0 dB high-pass filter was done using the diagram shown in Fig. 5.

Upon completion of the simulation, the PSpice magnitude response was plotted in Fig. 6. From this figure, it is clear that (i) Eq. (5) is in fact the transfer function of a third-order Chebyshev 1.0 dB low-pass filter; otherwise, the magnitude response would be different from that observed, and (ii) the theoretical values are very close to the simulated values given in the graph, as summarized in Table II.

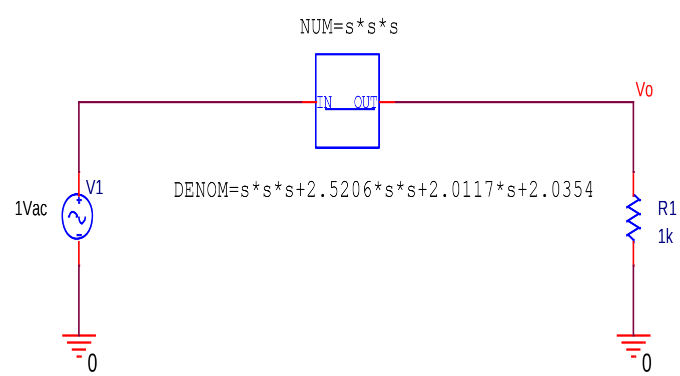

FIGURE 5. PSpice implementation of a third-order Chebyshev 1.0 dB high-pass filter with cut-off frequency of 1 rad/s.

TABLE II. Comparison of theoretical and PSpice simulated values.

Parameter	Theoretical Value	Simulated Value
f_{cutoff}	159.155 mHz	159.071 mHz
$f_{\min}$	318.33 mHz	318.58 mHz
$f_{\max}$	183.78 mHz	183.51 mHz
$\lvert T(\omega)\rvert_{\min}$	-1.0000 dB	-0.99981 dB
$\lvert T(\omega)\rvert_{\max}$	0 dB	$80.8017\text{x}10^{-6}$ dB

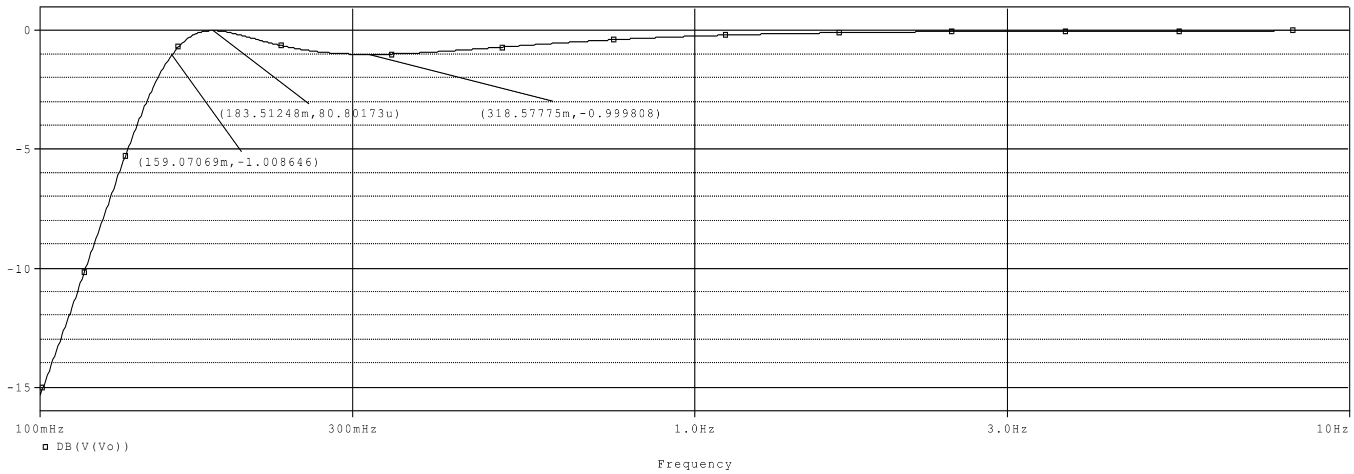

FIGURE 6. PSpice magnitude response (gain in dB) of a third-order Chebyshev 1.0 dB high-pass filter, as simulated with Fig. 5.

V. CONCLUSIONS

We have shown that the minimum and maximum gains of third-order low-pass and high–pass filters can be found without using calculus. The method uses the little known mathematical fact that the extrema of cubic functions can be found by purely algebraic methods. Furthermore, PSpice simulations were shown to verify the theoretical calculations.

REFERENCES

[1] Lennes, N. J., *Note on maxima and minima by algebraic methods*, The American Mathematical Monthly **17**, Issue 1, 9-10 (1910).
[2] Peterson, *Max and min of functions without derivative*, The Math Forum, April 22nd, 2003. http://mathforum.org/library/drmath/view/62730.html: accessed on 10th May, 2013.
[3] Cherkas, B., *Finding polynomial and rational function turning points in precalculus*, International Journal of Computers **8**, 215-234 (2003).
[4] Taylor Jr., R. D. and Hansen, R., *Optimization of cubic polynomial functions without calculus*, The Mathematics Teacher **101**, 408-411 (2008).
[5] Orcad PSpice demo download: http://www.cadence.com/products/orcad/pages/downloads.aspx#demo
[6] Stensby, J., "Chebyshev filters," http://www.ece.uah.edu/courses/ee426/Chebyshev.pdf : accessed on 10th May, 2013.
[7] "Practical filters," www.mrc.uidaho.edu/~dennis/ECE450-F11/8.5%20Practical.doc: accessed 10th May, 2013.

APPENDIX

Let (r_1, b) be the coordinate of one of the extrema of the cubic *f(x)* as shown in Fig. A1. Then, to find an extremum of the cubic *f(x)*, we seek the intersection of *f(x)* with the line *g(x) = b*, with *g(x)* tangent to *f(x)* at (r_1, b). If the line is to be tangent to the curve at (r_1, b), two of the three roots $(r_1, r_1 \text{ and } r_2)$ of *f(x)-b* must be coincident [1], as shown in Fig. A2. Hence,

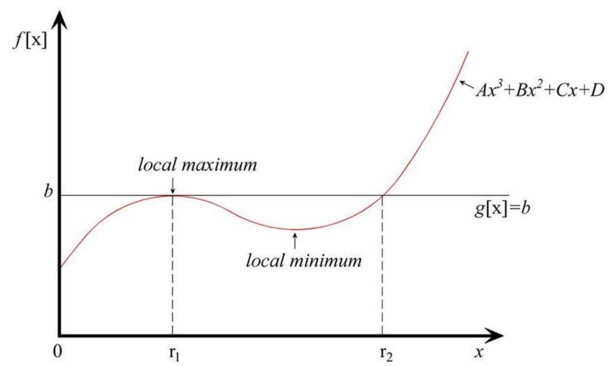

FIGURE A1. Plot of the function $f(x) = Ax^3 + Bx^2 + Cx + D$ intersecting with the line $g(x) = b$. Note that the line is tangent to the cubic function at the extremum point.

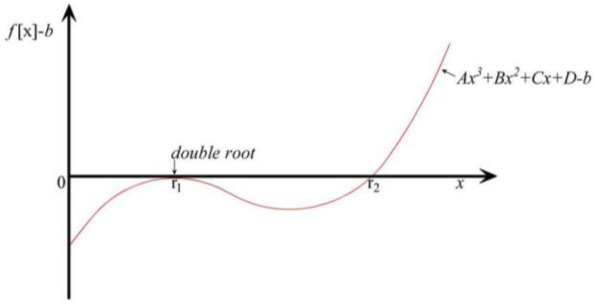

FIGURE A2. Vertical translation of the cubic function of Fig. A1. Note that there is now a double root of the translated cubic $f(x)-b$ at the extremum point.

$$\begin{aligned} Ax^3+Bx^2+Cx+D-b &= A(x-r_1)^2(x-r_2) \\ &= A(x^2-2xr_1+r_1^2)(x-r_2) \\ &= A\left[x^3-x^2(r_2+2r_1)+xr_1(2r_2+r_1)-r_1^2r_2\right]. \end{aligned} \tag{A1}$$

(Note that the necessity of $Ax^3+Bx^2+Cx+D-b=A(x-r_1)^2(x-r_2)$ was also proven by purely algebraic means in Eq. (1) of [3]).

Equating coefficients of Eq. (A1) gives:

$$-A(r_2+2r_1)=B, \tag{A2}$$

$$Ar_1(2r_2+r_1)=C, \tag{A3}$$

$$-Ar_1^2r_2=D-b. \tag{A4}$$

From Eq. (A2),

$$r_2=-\left(\frac{B}{A}+2r_1\right). \tag{A5}$$

Using Eq. (A5) in Eq. (A3) and simplifying yields the desired equation:

$$3Ar_1^2+2r_1B+C=0. \tag{A6}$$