# Motion through a velocity selector analyzed using a galilean transformation

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**Abstract**
The motion of a charged particle through a velocity selector is an example of a general class of motions through crossed electric and magnetic fields. In this paper particular attention is paid to the relationship between observables of this type of motion measured in two reference frames connected through a galilean transformation: the lab frame and a frame traveling at speed E/B relative to the lab ("selector frame"). The electromagnetic field only has a magnetic component in the selector frame, which makes solution of the equations of motion easier, and also affects conserved quantities such as energy and angular momentum. The electric field in this context breaks the rotational symmetry in one frame affecting angular momentum conservation consistent with Noether's theorem. Analyzing the motions in this manner is a way to introduce concepts from relativity into discussions of electromagnetism.

**Keywords:** Electromagnetism, Physics Education.

**Resumen**
El movimiento de una partícula cargada a través de un selector de velocidad es un ejemplo de una clase general de los movimientos a través de campos eléctricos y magnéticos cruzados. En este documento se presta especial atención a la relación entre los observables de este tipo de movimiento, medido en dos marcos de referencia conectados a través de una transformación de Galileo: el sistema del laboratorio y un marco que viaja a la velocidad de E/B relativo al laboratorio ("marco selector"). El campo electromagnético sólo tiene un componente magnético en el marco selector, lo que hace que la solución de las ecuaciones de movimiento sea más fácil; y también afecta a cantidades conservadas como la energía y el momento angular. El campo eléctrico en este contexto rompe la simetría de rotación en una trama que afecta a la conservación del momento angular consistente con el teorema de Noether. El análisis de los movimientos de esta manera es una forma de introducir los conceptos de la relatividad en las discusiones sobre el electromagnetismo.

**Palabras clave:** Electromagnetismo, Education en Física.

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## I. INTRODUCTION

A common example of the practical uses of magnetic fields is the velocity selector; a device which is usually a component of a mass spectrometer. Mass spectrometers are instruments found in many pure and applied science laboratories. When treated in physics textbooks [1, 2] the velocity selector is an example of how a combination of electric and magnetic interactions can be used to steer particles that differ from a particular velocity (the "selector velocity") away from the aperture of the spectrometer, thus reducing the number of variables that the trajectory of the particle depends on within the spectrometer to one: the mass.

The motion of a charged particle through a velocity selector is part of a more general class of motions; namely motions through crossed electric and magnetic fields. In an educational setting it can be used to distinguish between the two interactions, one of which is clearly dependent on velocity and the other which is independent of this variable.

The combination of these two interactions is usually referred to as the Lorentz force:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}\ , \qquad (1)$$

which describes the interaction of a particle with charge q moving with velocity $\mathbf{v}$ interacting with fields $\mathbf{E}$ and $\mathbf{B}$.

In the cases considered here we are interested in field configurations such as that described in Figure 1, where a uniform electric field is directed parallel to the y axis described by the vector $\mathbf{E} = E\hat{\mathbf{y}}$ and a uniform magnetic field is parallel to the z axis, and described by the vector $\mathbf{B} = B\hat{\mathbf{z}}$. Textbook discussions are mostly focused on the motion at the selector velocity, but examining the other trajectories can shed light on the nature of the two distinct interactions that collectively are thought of as electromagnetism.

In this paper we examine those trajectories by using a galilean transformation to what we will call the "selector frame", namely a reference frame moving at the selector velocity relative to the lab. An observer in this frame will not be able to detect an electric field, only a magnetic field will be present in an operational sense. The particle trajectories in this frame are all circular (or helical) and thus solving the equations of motion in this reference frame is much easier than it would be in the lab frame.

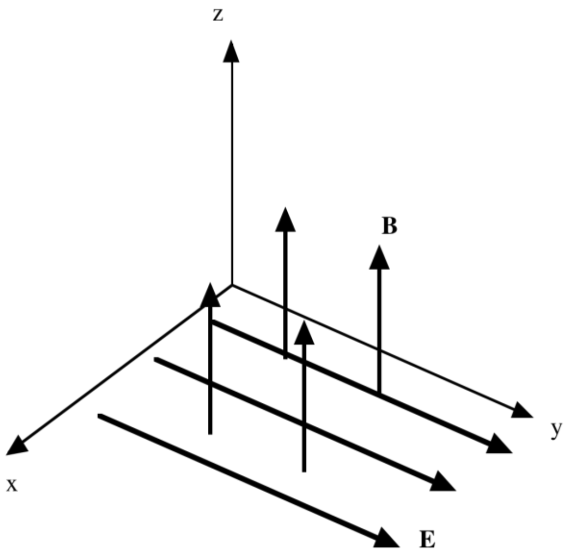


**FIGURE 1**. The coordinate system and field directions.

An inverse transformation can then be used to determine trajectories in the lab frame. This is of some practical mathematical use to be sure, but what is more interesting is the disappearance, if you will, of the electric field. This turns out to be a special case of the more general Lorentz transformations, and can be used to illustrate the relationship between electric and magnetic fields. In addition, conserved quantities differ from frame to frame. Mechanical energy is conserved in both frames, but the form differs. Angular momentum conservation, on the other hand, is directly affected by the presence (or absence) of an electric field which breaks the rotational symmetry consistent with Noether's theorem [3].

## II. TRAJECTORIES

According to Equation 1 the equations of motion for a charged particle of mass m and charge q moving in the *x-y* plane through the field configuration described in Figure 1 are given by:

$$m\frac{dv_x}{dt} = qv_y B. \tag{2a}$$

$$m\frac{dv_y}{dt} = qE - qv_x B. \tag{2b}$$

The form of Equation 2b suggests that a change of variable, that can be viewed as a galilean transformation, will simplify the mathematics, namely:

$$v'_x = v_x - v_s, \tag{3a}$$

which is consistent with:

$$x' = x - v_s t, \tag{3b}$$

where $v_s = \frac{E}{B}$ is the magnitude of the selector velocity.

Particles that enter the field region with velocity $\mathbf{v}_s = v_s\hat{\mathbf{x}}$ will follow straight line (inertial) paths at constant speed.

Applying Equations 3 to Equations 2, we find:

$$m\frac{dv'_x}{dt} = qv'_y B, \tag{4a}$$

$$m\frac{dv'_y}{dt} = -qv'_x B. \tag{4b}$$

Note that in this reference frame a charged test particle at rest will remain at rest. There is no electric field. Only a velocity dependent interaction will affect the test particle, namely a magnetic force. This result is consistent with the Lorentz transformation [4], for an observer moving along the x direction relative to the unprimed frame:

$$E'_y = \gamma(E_y - vB_z), \tag{5}$$

but must be viewed as a $v << c$ approximation as the Lorentz transformations also predict that the magnetic field will be different in this frame:

$$B'_z = \gamma(B_z - \frac{v}{c^2}E_y).$$

In the primed (selector) frame the velocity and position vectors consistent with Equations 4 are:

$$\mathbf{v}' = v'\cos(\omega t + \phi)\hat{\mathbf{x}} - v'\sin(\omega t + \phi)\hat{\mathbf{y}}, \tag{6a}$$

$$\mathbf{r}' = r_c\sin(\omega t + \phi)\hat{\mathbf{x}} + r_c(\cos(\omega t + \phi) + C)\hat{\mathbf{y}}, \tag{6b}$$

Where $\omega = \frac{qB}{m}$, and $r_c = \frac{mv'}{qB}$.

If we choose $\phi = 0$ and $C$=-1 Equations 6 describe the trajectory of a positively charged particle that enters the field region moving parallel to the *x* axis in the positive direction, and passing through the origin at *t*=0. This particle will execute half of a circular trajectory with clockwise rotation. If the field extends into quadrants 2 and 3 (or somehow just 3) the circle will be completed confined to quadrants 4 and 3 in the *x-y* plane. With suitable adjustments to the free parameters and the sign of the frequency, Equations 6 can describe motions for positive and negative particles (all circular) passing through the origin at $t$=0 parallel to the $x$ axis. For example, setting $C$=+1, $\phi=\pi$ and choosing a negative frequency, will describe the counter clockwise motion of a negatively charged particle, that passes through the origin along the $x$ axis, with a positive velocity.

Inverting the transformation described by Equations 3 we can determine solutions in the lab frame:

$$\mathbf{v}=(|\mathbf{v}-\mathbf{v}_s|\cos(\omega t+\phi)+v_s)\hat{\mathbf{x}} \\ -|\mathbf{v}-\mathbf{v}_s|\sin(\omega t+\phi)\hat{\mathbf{y}}, \tag{7a}$$

$$\mathbf{r}=(r_c\sin(\omega t+\phi)+v_s t)\hat{\mathbf{x}} \\ +r_c(\cos(\omega t+\phi)+C)\hat{\mathbf{y}}, \tag{7b}$$

In this frame:

$$r_c=\frac{|\mathbf{v}-\mathbf{v}_s|}{|\omega|}.$$

In Figure 2 we show several trajectories of positively charged particles with positive initial velocity as viewed in the lab frame. With sufficiently high initial velocity (relative to the selector velocity) the trajectories are cycloids, but at lower velocities they reduce to nearly sinusoidal shapes reducing further to a straight line motion at the selector velocity.

## III. CONSERVATION LAWS

A primary difference between an electric field and a magnetic field is that a magnetic field is incapable of doing mechanical work. The distinction between the two interactions in this regard is evident here, especially if we compare the two reference frames. In the primed, or selector frame, the net force on the particle takes the form: $\mathbf{F}'=q\mathbf{v}'\times\mathbf{B}$ and the force is always orthogonal to the velocity. As a result, no work is done and the kinetic energy of the particle is conserved. This is not true in the lab frame where an electric field is present and the net force takes the form of Equation 1. In this frame, the quantity (to within an arbitrary constant) $\mathrm{H}=\frac{1}{2}mv^2-qEy$ is conserved. In both cases we can refer to the conserved quantity as the mechanical energy, but in the primed frame we would describe the quantity by $H'=\frac{1}{2}mv'^2$. The galilean transformation preserves the energy conservation principle as it applies to mechanical energy, but the form is different.

The primary principle, that the quantity is conserved, is retained.

It is interesting to note that, the net force vector is invariant under the transformation described. The galilean transformation does not change the acceleration of the particle, both observers measure the same vector, thus the net force is the same in both frames. It is the relationship between the velocity and acceleration vectors that is different for the two observers. This affects the resulting motions, and the scalar quantities that are conserved.

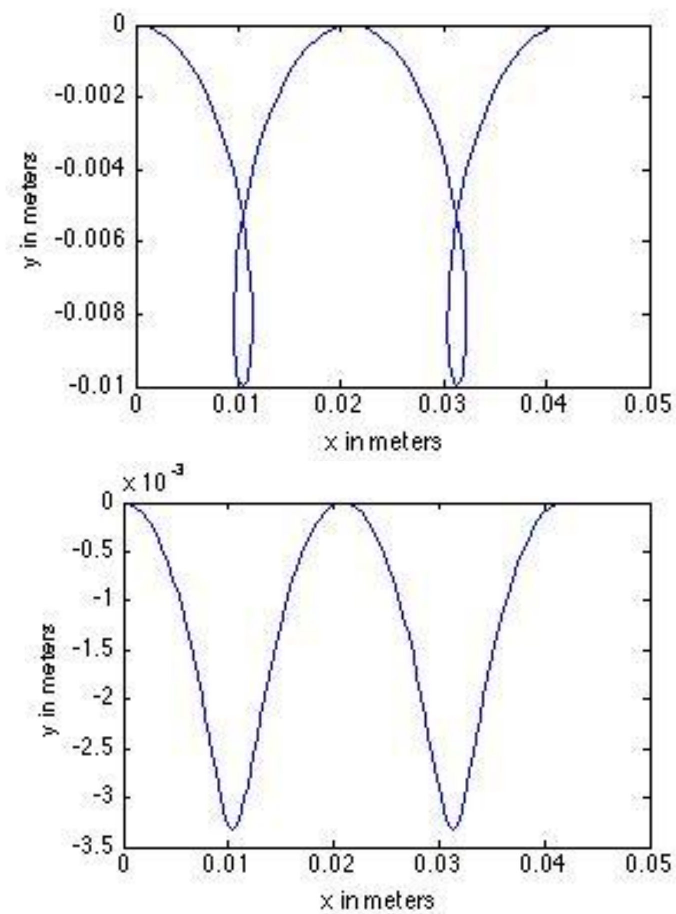


**FIGURE 2.** Two trajectories for a singly ionized positive ion (m=16 amu) entering the field region along the positive x axis at t=0. Top: $v_{x0}=2.5v_s$. Bottom: $v_{x0}=1.5v_s$. B=0.1 T, E=200 V/m.

We can also understand this using a lagrangian/ hamiltonian approach. Equation 2 can be derived from the lagrangian:

$$L=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+qBxv_y+qEy,$$

which can be written in the form:

$$L=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{d}{dt}(qByx)-qByv_x+qEy.$$

Since Hamilton's principle is invariant to a total time derivative added to the lagrangian:

$$L=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2-qByv_x+qEy, \tag{8}$$

is equivalent in that it will generate the same equations of motion. From Equation 8 using the generalized momenta $p_x = mv_x - qBy$ and $p_y = mv_y$, we can generate the hamiltonian:

$$H = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 - qEy,$$

which is a constant of the motion in the lab frame and can also be expressed in terms of the momenta and coordinates alone.

But Equation 8 can also be written in the form:

$$L = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 - qBy(v_x - v_s)$$

which suggests a transformation to the primed frame. Since galilean transformations do not affect time derivatives of velocities, we can add primes to the two kinetic energy terms, and write out a lagrangian (in the same form) that will generate the equations of motion in the primed frame:

$$L' = \frac{1}{2}mv_x'^2 + \frac{1}{2}mv_y'^2 - qByv_x' . \qquad (9)$$

The generalized momenta also have the same form in this frame:

$$p_x' = mv_x' - qBy' ,\ p_y' = mv_y'.$$

But now, combining these momenta with Equation 9 we generate the hamiltonian:

$$H' = \frac{1}{2}mv_x'^2 + \frac{1}{2}mv_y'^2,$$

where the absence of a potential energy term is a direct consequence of there being no electric field in this frame.

This quantity is a constant of the motion in the primed frame. In each frame the hamiltonian does not depend explicitly on time and is therefore a constant of the motion.

Noether's theorem [4] states that translational invarience of the lagrangian leads to conservation of linear momentum. The x component of the generalized linear momentum (which includes an electromagnetic term) is conserved in both frames, as the electric field does not break the translational symmetry of the system in that direction: in each frame the lagrangian is cyclic in the x coordinate.

We cannot say the same for rotational symmetry as the electric field does play the role of disrupter, in this regard, in the lab frame. This affects angular momentum as we discuss below.

Angular momentum is another quantity associated with a conservation law. In the primed frame it is fairly easy to see that an observer using an axis of rotation that passes through the center of one of the circular orbits, will find that the particle's angular momentum is conserved, namely:

$$\frac{d\mathbf{\Omega}'}{dt} = \mathbf{r}_c' \times q\mathbf{v}' \times \mathbf{B} = 0 \ , \qquad (10)$$

due to the centripetal nature of the force. In the lab frame the same result will be arrived at even though this observer explicitly takes into account the effect of the electric field:

$$\frac{d\mathbf{\Omega}}{dt} = \mathbf{r}_c \times q\mathbf{v} \times \mathbf{B} + \mathbf{r}_c \times q\mathbf{E} \cdot \qquad (11)$$

To see this note that in each frame the radial vector is the difference between the position vector of the particle and the position of the center of the circle, both relative to the respective origin: $\mathbf{r}_c = \mathbf{r} - \mathbf{r}_0$ and $\mathbf{r}_c' = \mathbf{r}' - \mathbf{r}_0'$. However, displacements are invariant under a galilean transformation so $\mathbf{r}_c' = \mathbf{r}_c$. Also, $\mathbf{v}_s = \frac{E}{B}\hat{\mathbf{x}}$, which means $\mathbf{E} = -\mathbf{v}_s \times \mathbf{B}$.

If we make this substitution into Equation 11 recognizing that $\mathbf{v}' = \mathbf{v} - \mathbf{v}_s$, we find:

$$\frac{d\Omega}{dt} = \mathbf{r}_c' \times q\mathbf{v}' \times \mathbf{B} = \frac{d\Omega'}{dt},$$

so, the angular momentum about the axis of one of the circular orbits is conserved in both frames. Following a moving axis, not surprisingly is equivalent to transforming to a moving frame of reference.

But what if each observer determines the angular momentum relative to the origin of their respective coordinate systems? In the primed and unprimed frames respectively we have:

$$\frac{d\mathbf{\Omega}'}{dt} = \mathbf{r}' \times q\mathbf{v}' \times \mathbf{B}, \qquad (12)$$

$$\frac{d\Omega}{dt} = \mathbf{r} \times q\mathbf{v} \times \mathbf{B} + \mathbf{r} \times q\mathbf{E}, \qquad (13)$$

and the position vectors are related by the transformation equation $\mathbf{r}' = \mathbf{r} - \mathbf{v}_s t$. Using this relationship in Equation 13 and the fact that $\mathbf{E} = -\mathbf{v}_s \times \mathbf{B}$ we find:

$$\frac{d\mathbf{\Omega}}{dt} = \frac{d\mathbf{\Omega}'}{dt} - qEv_x' t\hat{\mathbf{z}}. \qquad (14)$$

This is actually a somewhat general result, independent of where on the y axis, the axis of rotation is chosen. For example, if both observers shift the axis along their respective y axes to $y = y' = \pm r_c$, the primed observer will find that the angular momentum is constant as described by Equation 10. In the lab frame this quantity will change at a rate given by Equation 14 namely, $\frac{d\mathbf{\Omega}}{dt} = -qEv_x' t\hat{\mathbf{z}}$. This result can be understood in light of Noether's theorem. The electric field breaks the rotational symmetry about the z axis in the lab frame, and angular momentum is not conserved. In the primed frame, where there is no electric field, an observer can choose an axis of rotation about which angular momentum is conserved.

The symmetry differences are apparent even if we choose the origin as the axis. Since $\mathbf{r}' = \mathbf{r}'_c + \pm r_c \hat{\mathbf{y}},$ and considering Equation 10, we can calculate the derivative of the angular momentum in the primed frame about the origin:

$$\frac{d\Omega'}{dt} = \mp r_c q v'_y B \hat{\mathbf{z}},$$

for which the time average over a cycle vanishes, due to the sinusoidal character of the velocity. This is not the case in the lab frame due to the explicit time dependence of Equation 14. And in the primed frame, it is possible through a fixed origin shift $(0,0) \rightarrow (0, \pm r_c)$ to convert this vanishing time average to a vanishing instantaneous value.

## IV. DISCUSSION

Electric and magnetic interactions are distinct but intimately related. Like other measurable quantities they must be defined operationally, or in terms of how they are measured. This is where the idea of a test charge or in the case of magnetism, test current, comes into play. To an observer in the lab frame of reference, a charged particle travelling along the x axis, at the selector velocity has zero electromagnetic force acting on it; an observer moving at the selector velocity draws the same conclusion but the particle is at rest in this frame. The lab frame observer upon viewing particle motions with varying velocities will conclude that both electric and magnetic interactions are taking place and these interactions are simply balancing at the selector velocity. For the observer in the selector frame, no measurements will reveal the presence of an electromagnetic interaction that is not velocity dependent; hence, this observer cannot detect an electric field. In an operational sense this field is not present. If we were to argue that it is present but undetectable, we would be arbitrarily singling out the lab frame as some special reference frame. Doing so, would violate the basic principles of relativity (either galilean or einsteinian).

Thinking relatively is hard for students, as the idea of absolute motion is embedded into our common sense notions about the world. But the fact that magnetic interactions depend on velocity and electric interactions do not, is a hint that reference frames must play a role in electromagnetism. The field concept is also hard to grasp, and may seem like mere mathematical convenience when introduced in the context of electrostatics. Examples like the one analyzed here, where both interactions are in play and fluid, demonstrates the full power of Maxwell's theory.

## REFERENCES


[1] Serway, R. W. and Jewett, J. D., *Physics for scientists and engineers with modern physics,* 9th Ed. (Brooks/Cole, Boston, 2014).

[2] Young, H. D. and Friedman, R. A., *University physics with modern physics*, 13th Ed. (Pearson, San Francisco, CA, 2014).

[3] Deslodge, E. A. and Karch, R. I., *Noether's theorem in classical mechanics*, Am. J. Phys. **45**, 336-339 (1977).

[4] Feynman, R. P., Leighton, R. B. and Sands, M., *The Feynman lectures on physics. Vol. 2*, (Addison Wesley, Reading: USA, 1963).