Students' difficulties in problems that involve unit-vector notation

Pablo Barniol, **Genaro Zavala**
Physics Education Research and Innovation Group.
Department of Physics, Tecnológico de Monterrey, Campus Monterrey,
E. Garza Sada 2501, Monterrey, N. L. 64849 México

E-mail: pablo.barniol@itesm.mx, genaro.zavala@itesm.mx

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Abstract
Many researchers have analyzed students' understanding of vector concepts. However, at the moment there is a need in the physics education research area for a study that presents a complete analysis of students' difficulties with problems that involve unit-vector notation. This article responds to this need. We administered a test with open-ended problems to a total of 972 students who had completed the required introductory physics courses at a large private Mexican university. We found frequent errors when students sketched a vector written in the unit-vector notation and calculated direction, magnitude, sum, subtraction, scalar multiplication, dot product and cross product in problems that involve this notation. These errors persist even after instruction and have not been reported in the literature. The results of this research can help guide the development of new instructional materials that will attempt to improve students' understanding of problems involving unit-vector notation.

Keywords: vector, students' difficulties, unit-vector notation.

Resumen
Muchos investigadores han analizado la comprensión de los conceptos de vectores de los estudiantes. Sin embargo, en este momento hay necesidad en el área de la investigación en educación física, para un estudio que presente un análisis completo de las dificultades de los estudiantes con problemas que implican la notación unidad en vectores. Este artículo responde a esta necesidad. Administramos una prueba con problemas de composición abierta para un total de 972 estudiantes, quienes habían completado los cursos de introducción a la física requeridos para una gran universidad privada mexicana. Encontramos errores frecuentes cuando los estudiantes esbozaron un vector escrito en la notación de unidad-vector y dirección calculada, en magnitud, en suma, en resta, en multiplicación escalar, en producto escalar y en producto vectorial, en problemas que involucran esta notación. Estos errores persisten incluso después de la instrucción, no se han reportado en la literatura. Los resultados de esta investigación pueden ayudar a guiar el desarrollo de nuevos materiales de instrucción, que intentarán mejorar la comprensión de los estudiantes en los problemas que implican la notación en unidad-vectores.

Palabras clave: vector, dificultades de los estudiantes, notación de unidad-vector.

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I. INTRODUCTION

Vector concepts are very important in physics, since many physical quantities are defined as vectors. For example force, velocity and acceleration are defined as vectors, and electric and magnetic field are defined as vector fields.

Many researchers have analyzed misconceptions of these types of physical quantities [1], and some of them have described the relationship between these misconceptions and students' difficulties with vector concepts [2, 3]. Flores *et al*. [2] mentioned that after traditional instruction in mechanics many students lack the ability to reason about vectors that represent forces and kinematic quantities. Moreover, Saarelainen *et al*. [3] established that the absence of appropriate vector thinking in the case of the fields was the primary reason for the vague usage of the field concept in answering some electricity and magnetism conceptual problems.

Although many studies have centered on students' difficulties with vector concepts in problems with mathematic and physical context, only a few of them have focused on students' difficulties in problems that involve unit-vector notation. This is remarkable because this notation is commonly used in the introductory physics university courses. This article seeks to address this need.

Therefore, the objective of this study is to analyze the difficulties that students who have completed their introductory physics university courses still have with problems that involve the unit-vector notation.

II. PREVIOUS RESEARCH

The previous studies that analyze students' understanding of vectors concepts can be clustered in three groups:

1) studies that analyze students' understanding of vectors concepts in problems with no physical context [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15],

2) studies that investigate students' understanding in problems with a physical context [2, 3, 6, 7, 9, 15, 16, 17, 18, 19, 20], and

3) studies that compare students' performance on solving problems with physical context and problems with no physical context [6, 7, 9, 13, 15, 19].

Note that some of these studies pertain to more than one group.

The present study is classified within the first group since it is a report on students' understanding of problems with no physical context. This first group can also be divided in two subgroups: a) studies that focus on students' understanding of problems that involve unit-vector notation [4, 6], b) studies that focus on students' understanding of problems in which the vectors are presented in graphical form (the rest of the reports). From this subdivision, we can note that only two studies analyze students' understanding of problems with unit-vector notation. Next we describe briefly what is established in the two articles that focus on problems that involve unit-vector notation.

In the first report, Knight [4] designed an open-ended problem test, and administered it to students who entered the university before receiving any instruction in vectors.

He found that 50% of these students had no useful vector knowledge at all. In the designed test, four problems involve the unit-vector notation. The first problem asks the student to establish the y-component of the vector 3**i**-2**j**+3**k**.

Knight found two frequent errors:

1) students would exclude the negative sign and

2) they would include the unit vector in the answer (2**j**).

The second problem asks them to write, in unit-vector notation, the vector -3**i**+2**j** which is presented in graphical form. The author found that some students frequently made the error of writing the vector as $3x+2y$. The third problem asks for the addition of two vectors written in the unit-vector notation, and the fourth problem asks to calculate the dot product of two vectors written in this notation. In these two problems, the author didn't detect any frequent errors.

In the second work, Van Deventer [6] compared students' performance in isomorphic ("same form") problems with physical and no-physical context. The author carried out eleven interviews with university students who had taken the introductory physics courses, in order to identify the frequent errors. Three interviewee's problems involved unit-vector notation. In the addition and subtraction problems, the researcher didn't find any errors.

On the other hand, in the dot product the author found one error: some students believed that the dot product of vector 3**i** and vector 4**j** was12**ij**.

After this synthetic view of these two studies, we can see that so far, the studies that have focused on students' difficulties in problems that involve unit-vector notation are only partial. There are no other studies, to our knowledge; therefore, we believe that this vacuum justifies the present research.

Results of the present study were used, following the recommendations of Beichner [21], to build five items of our reliable 20-item multiple choice vector concept test (Test of Understanding of Vectors, TUV) that was presented briefly in a short paper [22]. The results of some of our other works presented elsewhere [8, 9, 10, 14, 15] were used to build the other items of the test that evaluate the students' understanding of vector concepts posed in graphical form.

Finally, some previous studies from this journal [23, 24] have presented new technology based instructional strategies to teach vector concept.

III. METHODOLOGY

This study was conducted at a large private Mexican university. In this institution the textbooks used in the physics courses are "Physics for Scientists and Engineers" by Serway and Jewett [25] and "Tutorials in Introductory Physics" by McDermott and Shaffer [26].

A. Design of the open-ended problems

We decided to use open-ended problems because they have been used to identify students' difficulties in vector concepts [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].To establish these problems, we asked physics faculty members in interviews about the vector properties and operations in this notation and after that, we arrived to a consensus of eight problems that are used in this study. In the Appendix of this article, we show these eight problems, in which students are asked to:

1) sketch a vector,
2) calculate the direction of a vector,
3) calculate the magnitude of a vector,
4) add two vectors,
5) subtract two vectors,
6) perform a negative scalar multiplication of a vector,
7) calculate the dot product of two vectors, and
8) calculate the cross product of two vectors.

Note that in the analysis of each of the problems we will describe briefly the background that the students must have to answer each of the items correctly.

B. Study Participants

The objective of the study allows us to choose the participants of this study. In this institution the students who are completing their introductory physics university courses are students who had finished the calculus-based "Electricity and Magnetism" course. There were a population of 972 students; however, we had some limitations. The first limitation was temporal since we were limited by the time it takes students for a diagnostic of open-ended problems. The second limitation was based on the fact that many of the problems that we wanted to administer were highly related to each other (for example: the dot product and the cross product, the addition and the subtraction, and the direction and the magnitude). Some authors (*i.e.* Gray [27]) have pointed that solving a problem when a related problem is solved previously could influence the answer.

Due to time limitations and to avoid any possible influence between related problems we decided to randomly divide the sample into six different groups, following the methodology by Barniol & Zavala [8].

The six groups of students were divided as follow: 143 students did Problem 1 (graphic representation of a vector), 197 did Problems 2 and 5 (direction of a vector and subtraction), 139 did Problem 3 (magnitude of a vector), 202 did Problem 4 and 6 (addition and scalar multiplication), 143 did Problem 7 (dot product) and 148 students did Problem 8 (cross product).

C. Analysis of students' answers

To analyze students' answers in the open-ended problems we use a codification process established by Hernandez Sampieri *et al*. [28]. According to these authors, this process consists in finding and naming similar answers, then listing them, and finally establishing the proportions of each answer. In the analysis of each of the problems we first codify the most common errors in a table and then in the analysis of these errors we describe the reasoning or procedures that lead students to make these errors.

IV. RESULTS

In the following subsections, we present the results of students' answers to each of the problems. In each case we present the most common errors and their distribution. Then we describe in detail what students did in each of the errors.

A. Graphical representation of a vector written in the unit-vector notation

We designed Problem 1 (shown in the Appendix) to analyze students' difficulties in sketching a vector written in the unit-vector notation.

Students are asked to sketch a vector in a grid with coordinate axis. The statement of this problem is: "Draw in the grid the vector **A**=-2**i**+3**j**". As mentioned before, 143 students answered this problem. To answer this problem correctly students should know that this vector have a negative *x*-component of two units and a positive *y*-component of three units. Therefore, they should sketch the vector from the origin (0, 0) to the point (-2, 3).

Table 1 shows the proportion of the answers given by the students, and Figure 1 shows the errors graphically. In Table 1, we notice that 81% of students correctly represented vector -2**i**+3**j**. We found three errors.

TABLE I. Answers given for Problem 1: Graphic representation of vector -2**i**+3**j**.

Answer	*%*
Correct	81%
Vector from (-2,0) to (0,3)	7%
Vector in point (-2,3)	6%
Vector from (-2,0) to (-2,3)	1%
Others	5%

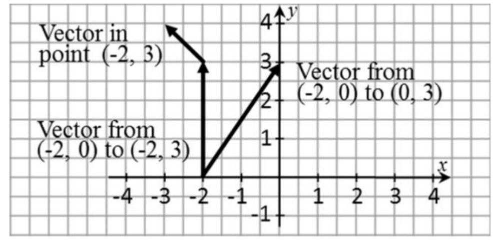

FIGURE 1. Frequent errors in problem 1.

1) Students sketched a vector from point (-2, 0) to point (0, 3).These students drew the components of the vector (from the origin in a tail-to-tail representation) and then sketched vector -2**i**+3**j** from the head of vector -2**i** to the head of vector 3**j**. This error seems to be related to the tail-to-tail error identified by Knight [4] in vector addition problems.

2) Students identified correctly the point (-2, 3), but instead of drawing vector -2**i**+3**j** from point (0, 0) to this point, students incorrectly sketched the vector at this point. The drawn vectors have different magnitudes and directions, but approximately one third of the students sketched a vector -**i**+**j** at this point (as shown in Figure 1).

3) Students drew a vector from point (-2, 0) to point (-2, 3). These students correctly identified these two points (-2, 0) and (-2, 3), which is one step of the correct procedure; however, they made the mistake of sketching the vector between these two points (as shown in Figure 1).

If we analyze these three errors as a group we note that they have in common that students identify points that are part of the correct procedure but make errors in the interpretation of the role of these points in the correct procedure.

B. Direction of a vector written in the unit-vector notation

To analyze students' difficulties with the direction calculation of a vector written in the unit-vector notation, we designed Problem 2 (shown in the Appendix) that asks students to calculate the direction of a vector. The statement of this problem is: "Consider vector **A**=-3**i**+4**j**. Calculate the direction of this vector measured from the positive x-axis."

As mentioned before, 197 students answered this problem. There are different procedures to answer this problem; however, a common correct procedure is that students visualize that vector **A** is in the third quadrant with its tail in the origin and with a negative *x*-component of three units and a positive *y*-component of four units. Then students realize that the direction of a vector is calculated as an angle in counterclockwise direction from the axis, *x*-axis, in this case. Students then calculate the angle between vector **A** and its x-component using the equation $\tan^{-1}(4/3)$ and finally they subtract this value to 180° obtaining an angle of 126.87°.

Table II shows the answers and distributions given by the students.

TABLE II. Answers given for Problem 2: Calculation of the direction of vector -3**î**+4**ĵ**.

Answer	%
Correct (126.87°)	17%
Presents only a correct sketch of the vector	18%
Presents only an incorrect sketch of the vector	6%
Considers that the direction is in the negative direction of x-axis and in the positive direction of y-axis	7%
Considers that the direction is in the negative direction of x-axis	7%
Magnitude of the vector (5)	5%
Unit vector in the direction of vector **A**	5%
-53.13°	5%
53.13°	4%
143.13°	3%
135°	1%
No answer	11%
Others	11%

As we can see in Table II, only 17% of the students correctly established that the direction of the vector is 126.87°. When we analyzed students' answers, we found that a considerable proportion (38%) of students didn't calculate the direction of vector **A** but presented a sketch of the vector or used a qualitative description when answering the question. We found that 24% of students presented a sketched of a vector, usually in a coordinate system: 18% presented a correct drawing of the vector and 6% of the students presented an incorrect drawing of the vector, such as the one shown in the analysis of the previous problem.

We also found that 14% used a qualitative description. 7% of the students answered that the direction of vector A is "in the negative direction of the x-axis and in the positive direction of the y-axis". Some students stated that explicitly, and some others referred to directions as "to the northwest" or "to the second quadrant". This incorrect conception was also detected in a previous report from us with a problem that asked students to identify (in a graphic) vectors with the same direction [8]. We also found that 7% of the students thought that the direction of vector -3**i**+4**j** (measured from positive x-axis) was in the negative direction of the x-axis.

The most frequent answers in this category were: "-3**i**", "3 units to the left", and "to the left".

It seems very probable that these students have difficulties with the phrase "measured from the positive x-axis", and believe that the requested direction is the one of the x-component of the vector. As noticed before, about 38% of students presented a sketch or used a qualitative description. It is interesting to mention that Aguirre and Erickson [16] also found in an interview-based study that, when students are asked to calculate the direction of a vector, a significant number of them used qualitative instead of quantitative descriptions when answering the question.

We also discovered that 7% of students incorrectly stated that the direction of vector **A** is 5, confusing direction with magnitude. Moreover, we found that 5% of students calculated the unit vector in the direction of vector **A**, as the direction of this vector. It seems that these students have difficulties in understanding the meaning of the unit vector (we will also see this difficulty in the next problem).

Finally, we detected four errors in the calculation of the correct angle:

1) Students (5%) incorrectly answered -53.13°. These students directly calculated the direction as $\tan^{-1}(4/3)$.

2) Students (4%) incorrectly established that the direction is 53.13°, calculating it as $\tan^{-1}(4/3)$.

3) Students (3%) answered 143.13°. These students calculated first a 53.13° angle (as $\tan^{-1}(4/3)$) and then incorrectly added a 90° angle to this value.

4) Students established that the direction of the vector is 135°. These students incorrectly believed that the vector makes an exact negative 45° angle with the negative x-axis.

C. Magnitude of a vector written in the unit-vector notation

We designed Problem 3 (shown in the Appendix) to analyze students' difficulties with the magnitude calculation of a vector written in the unit-vector notation.

The statement of this problem is this: "Consider vector **A**= 2**i**+2**j**. Calculate the magnitude of this vector." As mentioned before, 139 students answered this problem. To answer this problem correctly students should calculate the magnitude of vector **A** using the Pythagorean Theorem ($\sqrt{(2^2+2^2)}$) obtaining $\sqrt{8}$. Table III shows the answers and distributions given by the students.

We notice from Table 3 that 74% correctly answered Problem 3 as $\sqrt{8}$. These students correctly used the Pythagorean Theorem $A=\sqrt{(2^2+2^2)}$. We found four frequent errors:

1) Students unexpectedly concluded that the magnitude of vector **A** is the unit vector in the direction of vector **A** (2**i**/√8+2**j**/√8). As mentioned in the previous analysis, students seem to have difficulties in understanding the meaning of this vector.

2) Students calculated the magnitude of vector **A** as 4. These students made a calculation error, because they incorrectly calculated $\sqrt{(2^2+2^2)}$ as 4.

3) Students considered that the magnitude of the vector is 2. Some students applied the Pythagorean theorem incorrectly as $A=\sqrt{(2+2)}$.

4) Students believed that the magnitude of the vector was 8. Some of these students also incorrectly applied the theorem as $A^2=\sqrt{(2^2+2^2)}$.

TABLE III. Answers given for Problem 3: Calculation of the magnitude of vector 2**i**+2**j**.

Answer	%
Correct (√8)	74%
2**i**/√8+2**j**/√8	4%
4	3%
2	1%
8	1%
No answers	9%
Others	8%

D. Addition of vectors written in unit-vector notation

To analyze students' difficulties with the addition of vectors written in the unit-vector notation, we designed Problem 4 (shown in the Appendix). The statement of this problem is: "Consider vectors A=3**i**+4**j** and B=2**i**-3**j**. Find the vector sum **R**=**A**+**B**." As mentioned before, 202 students answered this problem. To answer this problem correctly students must add the x-components and the y-components of both vectors to obtain the components of the vector sum 5**i**+**j**. Table 4 shows the distribution of answers given by students.

TABLE IV. Answers given for Problem 4: Addition of vectors 3**i**+4**j** and 2**i**-3**j**.

Answer	%
Correct (5**i**+**j**)	88%
5**i**-**j**	7%
No answer	1%
Others	4%

In Table 4 we notice that 88% of the students correctly answered this problem. We only found one significant error (7%) in which students incorrectly answered 5**i**-**j**. These students correctly added the x-components of the vectors (5**i**), but incorrectly added the y-components (-**j**) since they included an incorrect negative sign for the answer.

E. Subtraction of vectors written in unit-vector notation

To investigate students' difficulties with the subtraction of vectors written in the unit-vector notation, we designed Problem 5 (shown in the Appendix). The statement of this problem is: "Consider vectors **A**=3**i**+4**j** and **B**=2**i**-3**j**. Find the vector difference **T**=**A**-**B**." As mentioned before, 197 students answered this problem. To answer this problem correctly students should subtract the x-component of vector **B** from the x-component of vector **A** and the y-component of vector **B** from the y-component of vector **A** obtaining the vector difference **i**+7**j**. Table 5 shows the answers and distributions given by students.

TABLE V. Answers given for Problem 5: Subtraction of vectors (3**i**+4**j**) – (2**i**-3**j**).

Answer	%
Correct (**i**+7**j**)	75%
i+**j**	8%
No answer	8%
Others	9%

Table 5 shows that 75% of the students answered this problem correctly. As in the addition problem, we found only one significant error (8%) in which students incorrectly answered **i**+**j**. These students correctly subtracted the x-components of the vectors (**i**), but incorrectly subtracted the y-components (**j**). Students added the y-components. It seems that the negative sign of the y-component of vector **B** confused them.

F. Scalar multiplication of a vector written in unit-vector notation

We designed Problem 6 (shown in the Appendix) to analyze students' difficulties with the scalar multiplication of a vector written in the unit-vector notation. The statement of this problem is: "Consider vector **A**=-2**i**+6**j**.

Find vector **T**=-3**A**." As mentioned before, 202 students answered this problem. To answer this problem correctly students should multiply the x- and y-components of vector **A** by -3 obtaining the vector 6**i**-18**j**. Table 6 shows the answers and distributions given by students.

TABLE VI. Answers given for Problem 6: Scalar multiplication -3(-2**i**+6**j**).

Answer	%
Correct (6**i**-18**j**)	87%
-6**i**+18**j**	4%
No answer	4%
Others	5%

We see from this table that 87% of the students answered Problem 6 correctly. In this problem we also found only one significant error (4%): students incorrectly answered -6**i**+18**j**. These students had difficulties with the negative signs since they found the correct absolute value of the components but incorrectly established the sign of both components.

It seems probable that these students performed the operation using a multiplier of +3 instead of -3.

G. Dot product of vectors written in unit-vector notation

To analyze students' difficulties with the dot product of vectors written in the unit-vector notation, we designed Problem 7 (shown in the Appendix). The statement of this problem is: "Consider vectors **A**=1**i**+3**j** and **B**=5**i**. Calculate the dot product (**A**·**B**)." As mentioned before, 143 students answered this problem. One way to answer this problem correctly is to calculate the dot product of the x-component of vector **A** with the x-component of vector **B** obtaining 5, *i.e.* the angle between these vector components is 0°, and to calculate the dot product of the y-component of vector **A** with the x-component of vector **B** which is zero because the angle between these vector components is 90°. Finally the answer is adding the results of the dot product which is 5. Table 7 shows the answers and distributions given by students.

TABLE VII. Answers given for Problem 7: Dot product (1**i**+3**j**)·5**i**.

Answer	%
Correct (5)	35%
5**i**	20%
5i+3j	14%
15k	6%
6i+3j	4%
-15k	3%
No answers	4%
Others	14%

Unlike the previous three problems, in Table 7 we see that a small proportion of students (35%) answered Problem 7 correctly. In this Problem we found five frequent errors:

1) Students (20%) answered 5**i**. These students correctly multiplied the magnitude of the x-components of the vectors but incorrectly added the **i**-unit vector to that answer.

2) Students (14%) established that the answer is 5**i**+3**j**. These students made the same error as in the previous problem to obtain the 5**i**, and then they also incorrectly added the y-component vector of vector **A** (3**j**).

3) Other students (6%) confused the dot product with the cross product answering 15**k**. The great majority of students established explicitly in their procedure that they added the **k**-unit vector, forming the cross product **j**× **i**. (Note that the cross product is actually -15**k**, so there is also a sign error).

4) Students (4%) added incorrectly both vector (6**i**+3**j**).

5) Students (3%) answered -15**k**. These students also confused the dot product with the cross product but without the sign error.

It is interesting to analyze as a group the five frequent errors found in this problem that sum up to 47%. We first note that all of these students have difficulties to interpret the result of a dot product as a scalar since all of the answers represented vectors. Moreover, students who answer 6**i**+3**j** or 5**i**+3**j** (an additionally 20%) incorrectly use additional procedures in this problem. About this issue it's worth noting that investigations that analyze students' difficulties in the interpretation of the dot product have found that students interpret the dot product of two vectors as a vector between both vectors [6, 10]. Finally, the students who answer 15**k** or -15**k** (9%) confuse the dot product with the cross product.

H. Cross product of vectors written in unit-vector notation

To analyze students' difficulties with the cross product of vectors written in the unit-vector notation, we designed Problem 8 (shown in the Appendix). The statement of this problem is: "Consider vectors **A**=1**i**+3**j** and **B**=5**i**. Calculate the cross product (**A**×**B**)." As mentioned before, 148 students answered this problem.

To answer this problem correctly students first calculate the cross product of the x-component of vector **A** with the x-component of vector **B** obtaining 0 because the angle between these vector components is 0°. Then students calculate the cross product of the y-component of vector **A** with the x-component of vector **B** obtaining -15**k** applying the right hand rule and because the angle between these vector components is 90°.

Finally students add these two results obtaining -15**k**. Table 8 shows the answers given by students.

TABLE VIII. Answers given for Problem 8: Cross product (1**i**+3**j**)×5**i**.

Answer	%
Correct (-15**k**)	61%
+15**k**	10%
5**i**+3**j**	5%
6**i**+3**j**	2%
5**i**+15**k**	2%
-15**j**	2%
No answers	3%
Others	15%

We note from this table that 61% of students correctly answered Problem 7 (-15**k**). There were five errors. 1) The most frequent error (10%) was to omit the negative sign in the answer (+15**k**). 2) Students (5%) answered 5**i**+3**j**. As in the dot product problem, these students multiplied the magnitudes of the x-components of the vectors, including the **i**-unit vector in this answer (5**i**) and added the y-component vector of vector A (3**j**). 3) Students (2%) added the vectors (6**i**+3**j**). 4) Students (2%) answered 5**i**+15**k**.

These students obtained the 5**i** as did the students of the second error, and obtained the 15**k** of their answer, performing a cross product of the y-component of vector **A** (3**j**) with the x-component of vector **A** (5**i**). (Note that they also made a sign error in the cross product). 5) Finally, a few students (2%) gave -15**j** as the answer. As we can see, these students included the **j**-unit vector instead of the **k**-unit vector.

It is interesting to analyze as a group the five frequent errors found in this problem that sums up to 47%. We first note that the most frequent error +15**k** is a "sign error".

Scaiffe and Heckler [29] mention that this "sign error" is due to confusion about the execution and choice of the right-hand rule and lack of recognition that the cross product is no commutative. Moreover, the students who answer 6**i**+3**j** and 5**i**+3**j** (7%) incorrectly use additional procedures in this problem.

V. DISCUSSION

In the following section, we perform several analyses. First, we perform an overview analysis of students' difficulties with the eight problems. Then, we classify these difficulties according to their type. Finally, we compare our results with previous research studies.

A. Overview analysis of students' difficulties

First we perform an overview analysis of students' difficulties in the eight problems. We cluster the eight problems used in this study according to their difficulty level. The three most difficult problems are: Calculation of the direction of a vector (Problem 2, 17%), calculation of the dot product of two vectors (Problem 7, 35%), and calculation of the cross product of two vectors (Problem 8, 61%).

We note that the proportions of the correct answers of these problems are less than 70%. Therefore, students' difficulties are concentrated in these three problems. On the other hand, the rest of the problems show a low difficulty level with a correct proportion that is higher than 70%: Problems 1, 3-6.

B. Classification of students' difficulties by type

We classified the students' difficulties by type: (1) difficulties with the graphical interpretation of vectors, (2) difficulties with operational manipulation of vectors, and (3) difficulties with the interpretation of the operation.

B1. Difficulties with the graphical interpretation of vectors

There is only one problem (Problem 1) that explicitly asks students to make a graphical interpretation of a vector in unit-vector notation. However, in Problem 2, students also used graphs to try to answer the question, so we can analyze these difficulties in both problems.

First we analyze Problem 1. 19% of students answered this problem incorrectly. The two most common errors (that add up to 13%) are important to note, since without being able to make a graphical interpretation of a vector written in unit-vector notation correctly, students will not be able to solve problems in other advanced courses where they are required to use a mental view of vectors representing a physical quantity.

In the first error, students thought that the vector -2**i**+3**j** goes from the x-axis (point -2, 0) to the y-axis (point 0, 3), and in the second case they thought that the vector -2**i**+3**j** is a vector with its tail in point (-2, 3). In both mistakes, students correctly identified the points that needed to be recognized in the process in order to correctly draw the vector -2**i**+3**j**, however they didn't have the necessary knowledge to be certain about where the vector must start and end.

In Problem 2 we also detected that 6% of the students presented incorrect sketches of the required vector similar to the one analyzed in Problem 1. In this problem, we also observed other difficulties with the graphical interpretation of the direction concept of a vector. The first difficulty is that a considerable proportion of students (14%) don't interpret the direction of a vector as an angle; they think of it only as a description *(i.e*. "in the negative direction of the x-axis", "negative x-direction and positive y-direction".)

The second difficulty is with their comprehension of the graphical representation of the angle's general characteristics when measured from the positive x-axis. In the analysis of Problem 2, we observed that 9% of students incorrectly calculated the angle as ±53.13°. From a quick visualization of the requested vector -3**i**+4**j**, we can establish that the direction of this vector has to be between 90° and 180°.

Therefore, it is clear these students don't know or are unable to apply this general notion of the direction of a vector.

B2. Difficulties with operational manipulation of vectors

The majority of the problems (problems 3 to 8) ask students to make calculations of vectors in the unit-vector notation.

For these problems, students might only need to remember the definition of the mathematical operations to answer the problems correctly. It seems however, that in some cases the interpretation of the calculations plays a role in students' answers. In this section we analyze the difficulties with operational manipulation of vectors and in the next section we study the difficulties with the interpretation of the operation.

In problems 4 (addition of vectors), 5 (subtraction of vectors) 6 (scalar multiplication of a vector) most students respond to the problems correctly and their answers do not show many problems with interpretation. In these three problems the main mistakes come from sign errors. These mistakes seem to be more algebraic in nature than due to vector-related problems. In both problems (the subtraction and multiplication by a scalar, for instance), students made the mistake of not multiplying the negative sign on the operation to the second term of the two components.

In Problem 3 (magnitude of a vector) 5% of the students calculated the magnitude of the vector 2**i**+2**j** as 2, 4 or 8, incorrectly applying the Pythagorean Theorem. These mistakes seem to also be more caused by operational calculation errors than interpretation mistakes. However, in Problem 3 as well as in Problems 7 and 8, both types of errors might play a role in the results. In Problem 7, there are mistakes in which the dot multiplication of vectors **A**=1**i**+3**j** and B=5**i** can be thought of as calculation problems, since some students multiply the components but leave the unit vectors. This also happens in Problem 8, in which some students answered in a similar way, cross multiplying components and leaving the same unit vectors.

Students seem to have problems learning the rules of the dot and cross multiplications. However, as mentioned above, interpretation issues might play a role and we discuss this in the following subsection.

B3. Difficulties with the interpretation of the operation

In Problems 2 (direction of a vector), 3 (magnitude of a vector), 7 (dot product) and 8 (cross product) we find that there are definitely interpretation problems. In Problem 3, although most students answer this question correctly, there are some students (4%) who calculate the unit vector in the direction of the given vector as the magnitude of the vector.

We believe that this mistake is related to misconceptions related to a confusion between direction and magnitude. The evidence for this is that not only 4% of students calculate the unit vector when they are asked to calculate the magnitude, but also 7% of students calculate the magnitude of the vector when they are asked in Problem 2 to calculate the direction of the vector.

Another clear problem of interpretation is what occurs in Problem 7 (dot product) and Problem 8 (cross product). In Problem 7, students are asked to calculate the dot product between two vectors. They have to understand that the result should be a scalar. However, 47% of students answered with a vector instead. This is clear evidence that they do not understand the interpretation of a dot product.

Moreover, they not only confuse the dot product with the cross product (9% in Problem 7 and 7% in Problem 8) but also confuse these operations with the addition of vectors (4% in Problem 7 and 2% in Problem 8). These examples clearly show that some of the students' difficulties are the result of their problems with the interpretation of the operation.

B4. Comparison of the results with previous research

Finally, we compare our results with those of previous researchers. In the previous research section, we saw that no study had analyzed students' difficulties with problems similar to Problems 1 to 6 and 8. This is the first analysis of students' answers to these types of problems. We also observed that only Problem 7 (dot product problem) is related to an open-ended problem designed by Van Deventer [6], and it is the only one administered in interviews. As mentioned before, the only error detected in his study was the incorrect calculation of the dot product of vectors 3**i** and 4**j** as 12**ij**. It is interesting to note that we didn't find this error in our analysis of Problem 7 (Table 7).

We believe that this could be due to the differences in the samples analyzed. Van Deventer's sample included students that had finished their first introductory physics course. In our study we analyzed the responses of students who had finished their third introductory university physics courses.

We believe that the error "12**ij**" reported before is a lower cognitive level error which advanced students typically do not make. It's worth noting that our analysis of Problem 7 is the first analysis that presents a complete taxonomy of students' difficulties in an open-ended dot product's calculation problem.

VI. CONCLUSSIONS

In this study, we present a complete analysis of frequent errors in problems that involve unit-vector notation by students who have completed all of their introductory university physics courses. McDermott [30] established a process to increase students' understanding developing new instructional material based on research. In the first stage the researchers need to identify students' conceptual difficulties in a specific concept. Then in the second stage the researches use these finding to guide the development of new instructional material. The present study is an investigation of student understanding (first stage of the cycle) in problems that involve unit-vector notation.

In the Results section of this article, we detected and described frequent errors that the students had in the eight problems included in this study. It's important to mention that these difficulties persist in the students' responses, even after having completed three introductory university physics courses. Next we establish the most frequent error in each of the eight problems.

In Problem 1 we found that the most frequent error is to incorrectly sketch vector -2**i**+3**j** from the point (-2, 0) to the point (0, 3).

In Problem 2 (direction's calculation) we found that a significant proportion of students consider that the direction of vector -3**i**+4**j** measured "from the positive x-axis" is "in the negative direction of x-axis", and in the calculation of an angle, we found that the most common error is to state that the direction of this vector is -53.13°.

On the other hand, in Problem 3 (magnitude's calculation) we detected that the most frequent error is to believe that the unit vector in the direction of vector **A** is the magnitude of vector **A**.

In Problems 4-6 (addition, subtraction and negative scalar multiplication of a vector) we found difficulties with the signs in the calculations.

Finally, in Problem 7 (dot product's calculation) we discovered that the most frequent error is to calculate the dot product of vectors **A**=1**i**+3**j** and **B**=5**i** as 5**i**, and in Problem 8 (cross product's calculation) it is to report the incorrect sign for the answer.

These frequent errors have not been previously reported in the literature.

In the Discussion section, we first performed an overview analysis and a classification of the difficulties. In the overview analysis of the eight problems, we observed that the three most difficult problems were: Calculation of the direction of a vector (Problem 2, 17%), calculation of the dot product of two vectors (Problem 7, 35%), and calculation of the cross product of two vector (Problem 8, 61%).

Then, in order to understand the types of difficulties that students have with unit-vector notation, we classified those difficulties according to their type. The analysis showed that students have difficulties with the graphical interpretation of vectors when they are explicitly asked to make a sketch or when the solution of the problem leads them to do so.

Students also have difficulties with mathematical operations of vectors in which sometimes they forget the rules of the operations or they carry their difficulties with algebra into the vector operations. Others have difficulties with the interpretation of the operations because they confuse magnitude with direction or dot product with cross product.

Finally, it is important to mention that following the process to increase students' understanding established by McDermott [30], the results of this research can help to guide the development of new instructional materials (second stage of the cycle) that attempt to increase students' understanding of problems that involve unit-vector notation.

This development must explicitly address the most frequent error in each of the problems and focus especially on the three problems that are most difficult for the students (calculation of the direction of a vector, calculation of dot product, and calculation of cross product).

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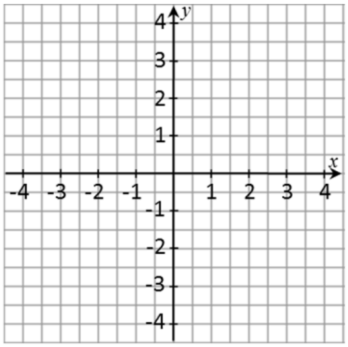
APPENDIX

The eight problems used in this study:
1. Draw in the grid the vector **A**=-2**i**+3**j**.



2. Consider vector **A**=-3**i**+4**j**. Calculate the direction of this vector measured from the positive x-axis.

3. Consider vector **A**= 2**i**+2**j**. Calculate the magnitude of this vector.

4. Consider vectors **A**=3**i**+4**j** and **B**=2**i**-3**j**. Find the vector sum **R**=**A**+**B**.

5. Consider vectors **A**=3**i**+4**j** and **B**=2**i**-3**j**. Find the vector difference **T**=**A**-**B**.

6. Consider vector **A**=2**i**+6**j**. Find vector **T**=-3**A**.

7. Consider vectors **A**=1**i**+3**j** and **B**=5**i**. Calculate the dot product (**A**·**B**).

8. Consider vectors **A**=1**i**+3**j** and **B**=5**i**. Calculate the cross product (**A**×**B**).