

Modelling the flight characteristics of a soccer ball



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Abstract

During a football game the ball is airborne a large percentage of the time. For this reason it is important to understand to principles that govern the motion of the ball in flight. This paper explores the various equations associated with the movement of the ball in flight. Numeric techniques are utilized to solve some of the equations and represent them graphically. The parameters in the model are manipulated to show how the graphs vary and agree with real life results.

Keywords: Mathematical modelling, drag coefficient, flight of ball, trajectory, velocity of ball.

Resumen

En un partido de fútbol la pelota está en el aire durante un gran porcentaje del tiempo. Por esta razón, es importante entender los principios que gobiernan el movimiento de la bola en vuelo. Este artículo explora las diversas ecuaciones asociadas con el movimiento de la pelota en vuelo. Se utilizan técnicas numéricas para resolver algunas de las ecuaciones y representarlas gráficamente. Los parámetros en el modelo son manipulados para mostrar cómo los gráficos varían y están de acuerdo con los resultados de la vida real.

Palabras clave: Modelos matemáticos, coeficiente de resistencia, vuelo de la bola, trayectoria, velocidad de la bola.

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I. INTRODUCCIÓN

Newton's second law of motion $F=ma$ where F is a force, m is a mass, and a is the acceleration is used to determine the flight of a ball. In the general case there are three forces acting on the ball, the force of gravity and two forces arising from the interaction with the air. For the latter of the two forces due to the air the drag acts in a direction opposite to the ball's velocity. The other force is the Magnus force which acts in the presence of spin and deflects the ball perpendicular to the velocity and to the axis of spin. If the spin axis of the ball is parallel to the ground, the Magnus force can provide lift. However, if the spin axis of the ball is perpendicular to the ground, the ball is made to bend flight.

We begin by considering the case where the effects of air are negligible. There is no horizontal force acting on the ball therefore, the equation for the horizontal velocity u is:

$$m \frac{du}{dt} = 0.$$

Since there is no horizontal force acting on the ball the horizontal component of velocity is constant, and u is equal to the initial horizontal velocity u_0 . The horizontal displacement, x is therefore:

$$x = u_0 t. \quad (1)$$

The equation for the vertical component of velocity v , is:

$$m \frac{dv}{dt} = -mg,$$

where g is the acceleration due to gravity. This equation has the solution:

$$v = v_0 - gt,$$

where v_0 is the initial vertical velocity. The velocity v at time t is obtained from:

$$v = \frac{dy}{dt}.$$

Therefore, the vertical displacement is obtained by integrating:

$$\frac{dy}{dt} = v_0 - gt,$$

to get:

$$y = v_o t - \frac{1}{2} g t^2. \quad (2)$$

Using equation (1) to eliminate t in equation (2) gives the equation for the trajectory:

$$y = \frac{v_o}{u_o} x - \frac{1}{2} \frac{g}{u_o^2} x^2. \quad (3)$$

This is the equation of a parabola.

The range R , of the flight is obtained by putting $y = 0$ in equation (3) since the ball starts off at $y = 0$ and returns to the ground. On doing this gives for the range,

$$R = \frac{2v_o u_o}{g}. \quad (4)$$

The time of flight is given by the time $t=T$, at which the displacement y returns to zero. From equation (2) this is given by:

$$T = \frac{2v_o}{g}.$$

Suppose that the initial angle between the initial trajectory and the ground is θ_0 , then:

$$v_o = V_0 \sin \theta_0 \quad \text{and} \quad u_o = V_0 \cos \theta_0, \quad (5)$$

where the initial velocity V_0 is given by:

$$V_o^2 = v_o^2 + u_o^2.$$

In terms of V_0 and θ_0 , the range given by equation (4) is:

$$R = \frac{2V_o^2 \sin \theta_0 \cos \theta_0}{g},$$

which is:

$$R = \frac{V_o^2 \sin 2\theta_0}{g}.$$

It is known that $\sin 2\theta_0$ has a maximum value at $\theta_0 = 45^\circ$, this angle gives the maximum range for a given V_0 ,

$$R_{\max} = \frac{V_o^2}{g}$$

Our attention now shifts to the drag. The equation for such is given by:

$$F_d = \frac{1}{2} C_D \rho A V^2. \quad (6)$$

Where the drag coefficient C_D depends on the velocity, is ρ the density of air, V is the velocity of the ball and A is its

cross-sectional area, in our case πa^2 , which is the maximum cross-section, where is the ball's radius.

Equation (6) as simple as it appears gives rise to associated equations of motion that are rather involved.

This is due to the fact that the velocity depends on C_D and also the drag force couples the equation for the horizontal and vertical components of the velocity. Newton's equations now become:

$$m \frac{du}{dt} = -F_d \cos \theta. \quad (7)$$

And:

$$m \frac{dv}{dt} = -F_d \sin \theta - mg. \quad (8)$$

where θ is the angle between the trajectory and the ground at time t , given by:

$$\tan \theta = \frac{v}{u}. \quad (9)$$

II. METHODOLOGY

It is important to note that equation (7) to (9) do not have an algebraic solution, but can be solved numerically. For simplicity, C_D is taken to be constant during the flight. For our case we use $C_D = 0.2$ since it is highly dependent on the Reynolds number. Wind tunnel testing has determined various drag coefficients for different shapes. For a sphere, it ranges from 0.07 to 0.5. A smooth sphere is around 0.5 and a rough sphere is around 0.07. A football's surface can be described somewhere in between these values. (www.grc.nasa.gov/www/k-12/airplane/shaped.html).

Using $v_1 = V \sin \theta$ and $u_1 = V \cos \theta$ equations (7) and (8) can now be written as:

$$\frac{du_1}{dt} = -\alpha u_1 V, \quad (10)$$

$$\frac{dv_1}{dt} = -\alpha v_1 V - g. \quad (11)$$

Where:

$$V^2 = v_1^2 + u_1^2. \quad (12)$$

And:

$$\alpha = \frac{1}{2} C_D \rho \frac{A}{m}.$$

Equation (10) to (12) will be solved with $C_D = 0.2$.

The density of air taken to be 1.2 kg m^{-3} at a temperature of 20°C the mass of the ball approximately 0.3969 kg m^{-3} and its cross sectional area is 0.39 m^2 , giving the value $\alpha = 0.015 \text{ m}^{-1}$. (The Science of Soccer, John Wesson). By solving for u and v we can integrate u and $\frac{dy}{dt} = v$ to obtain x and y respectively.

By manipulating V_0 and θ_0 we obtain numerical simulations for the flight path of the ball. However, our values for v_0 and θ_0 must be carefully chosen. Most medium to long range aerial passes in football are kicked with velocities ranging from 15m/s to 30m/s. Usually a 25m to 30m free kick is struck with a velocity of about 25m/s. This enables about 1 second to elapse between contact with the ball entering the goal. (www.soccerballworld.com/Physics.htm). Since our analysis is primarily concerned with aerial passing during play at medium to long range distances 20m to 70m, velocities of 15m/s to 30m/s is a good standard. One objective of the player is to flight the ball with just the right velocity to allow a teammate to collect and control. If too much velocity is applied this may be difficult.

Other forces can play a pivotal role in the movement of the football in flight. Let us consider the effect of wind moving parallel to the field's plane having speed w along a parabolic path. The equations of motion would take the form:

$$\frac{du}{dt} = -\alpha(u-w)V, \quad (13)$$

$$\frac{dv}{dt} = \alpha vV - g, \quad (14)$$

with V given by:

$$V^2 = (u-w)^2 + v^2. \quad (15)$$

Notice that the wind only affects the equation that relates to the horizontal velocity component since it can act in such a way to either speed up the motion or slow it down. We will assume that $+w$ corresponds to a trailing wind and $-w$ corresponds to a headwind.

Again we rely on numerical methods to solve these equations. It is interesting to note if we make the transformation $u-w \rightarrow u'$ and $v \rightarrow v'$, equations (13) to (15) take the form of equation (10) to (12) which we are already familiar with. We can see that u and v are replaced by u' and v' . If the equations are solved for u' and v' and u' and v' are calculated from $\frac{dx'}{dt} = u'$ and $\frac{dy'}{dt} = v'$, then the required solution can be achieved using the inverse transformations:

$$\begin{aligned} u &= u' + w, & v &= v', \\ x &= x' + wt, & y &= y'. \end{aligned}$$

We now present numerical simulations for both the horizontal and vertical displacement of a football when the angle to the horizontal is fixed and the initial velocity of the ball is varied between 15m/s and 30m/s. This is done in the case of a trailing wind and a headwind.

Furthermore, we fix the velocity and vary the initial angle of the ball between 10° and 45° . In each instance, we used wind speeds of 4m/s and 7m/s. Wikipedia has

Modelling the flight characteristics of a soccer ball classified these wind speeds as ranging from light breeze to moderate breeze. An Olympic sprinter running with a trailing wind of 4m/s is said to be wind aided and the time clocked is not recorded as legal. (International Athletics Federation). This is to say that a football moving through the air with winds speeds in this range will strongly influence its behaviour.

Consequently in addition to the wind having a velocity w moving parallel to the field's plane, the wind can act along the sides of the ball. The velocity is still taken as w but the direction is denoted by z . The motion of the velocity in this direction is denoted by V . The equation for the change in velocity in this direction with respect to time is:

$$\frac{dv_z}{dt} = -\alpha(v_z - w)V, \quad (16)$$

with:

$$V^2 = u^2 + v^2 + (v_z - w)^2.$$

Solving for α yields $0.015m^{-1}$ the above equation can be solved numerically. However, a simple procedure gives a formula for the sideways deflection of the ball's trajectory which is accurate for most cases. The equation for the forward motion is:

$$\frac{du}{dt} = -\alpha u V. \quad (17)$$

Dividing equation (16) by equation (17) gives:

$$\frac{dv_z}{du} = \frac{v_z - w}{u}. \quad (18)$$

Integrating equation (18) gives the solution:

$$v_z = w \left(1 - \frac{u}{u_0} \right), \quad (19)$$

where u_0 is the initial value of u and $v_z = 0$ initially.

The deflection z is obtained by solving:

$$\frac{dz}{dt} = v_z.$$

Hence, by using equation (19) for:

$$z = w \left(t - \frac{\int_0^t u dt}{u_0} \right).$$

The deflection d , over the full trajectory is therefore:

$$d = w \left(T - \frac{R}{u_o} \right),$$

where T is the time of flight and R is the range. It is important to observe that T and R are slightly affected by the side wind. A good approximation for d is obtained using their values with no wind.

So far, we haven't considered what transpires when spin is applied. A force known as the Magnus effect describes such motion. When a football is spinning the deflection caused by the Magnus effect displaces the ball in a perpendicular direction to the spin axis. This is to say that, if the spin axis is horizontal, the Magnus effect provides back lift and if the spin axis is vertical, the Magnus effect causes the ball's trajectory to bend sideways.

With medium and long range kicking we are primarily concerned with getting the ball to lift up high enough to clear opposing players and travel the required distance. Back lift produces this type of motion. Conventionally this force F_L is given by:

$$F_L = \frac{1}{2} C_L \rho A V^2 \quad (20)$$

by analogy with the drag force given in equation (6) This formula has its origin in aeronautics and the subscript L stands for the lift which would occur, for example, on a wing. For our purpose this expression is somewhat misleading because C_L depends on both the spin and the velocity.

For a spinning ball C_L is proportional to $q a / V$ provided that $q a / V$ is not too large and it is, therefore, convenient to write:

$$C_L = \frac{q a}{V} C_s,$$

where p is the angular frequency of the spin and is the radius of the ball, then:

$$F_L = \frac{1}{2} C_s \rho A a q V \quad (21)$$

Substituting for the air density $\rho = 1.2 \text{ kg m}^{-3}$, the radius $a = 0.11 \text{ m}$ and the cross-sectional area $A = 0.039 \text{ m}^2$ equation (21) becomes.

$$F_L = 2.6 \times 10^{-3} C_s q V \quad (22)$$

This sideways force produces a curved trajectory and the force is balanced by the centrifugal force $m V^2 / R$, where R is the radius of curvature of the trajectory. Using equation (22) with the mass of 0.3969 kg, the resulting radius of curvature is:

$$R = 153 \frac{V}{C_s q} \quad (23)$$

If we measure the rotation by the number of revolutions per second, f , then since $f = q / 2\pi$ equation (23) becomes:

$$R = 24 \frac{V}{C_s f} \quad (24)$$

It's more natural to think in terms of sideways displacement of the ball as illustrated in figure 1 below. If we approximate by taking the trajectory to have a constant curvature then using Pythagoras's equation:

$$L^2 + (R - D)^2 = R^2,$$

and taking $D \ll R$ so that D^2 is negligible:

$$D = \frac{L^2}{2R}.$$

Using equation (24) this becomes:

$$D = \frac{C_s L^2 f}{48 V} \text{ metres } V \text{ in } \text{m s}^{-1} \quad (25)$$

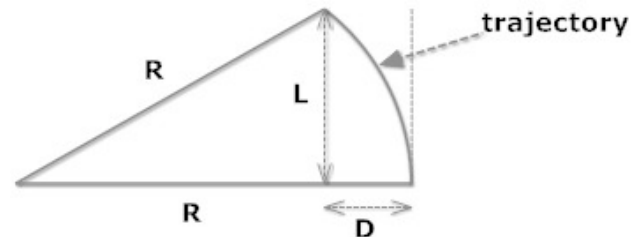


FIGURE 1. Mathematical construction for the trajectory.

The time of flight is L/V and so the number of revolutions of the ball during its flights is $n = Lf/V$. Substitution of this relation into equation (25) gives:

$$\frac{D}{L} = C_s \frac{n}{48}.$$

We have no direct measurement of C_s for footballs but experiments with other spheres have given values in the range 0.25 to 1 depending on the nature of the surface.

Taking $C_s = 0.5$ we obtain the approximate relation:

$$\frac{D}{L} = \frac{n}{100} \quad (\text{The Science of Soccer John Wesson}).$$

For example, a deviation of 1 m over a length of 30 m would required the ball to undergo about 3 revolutions.

III. RESULTS

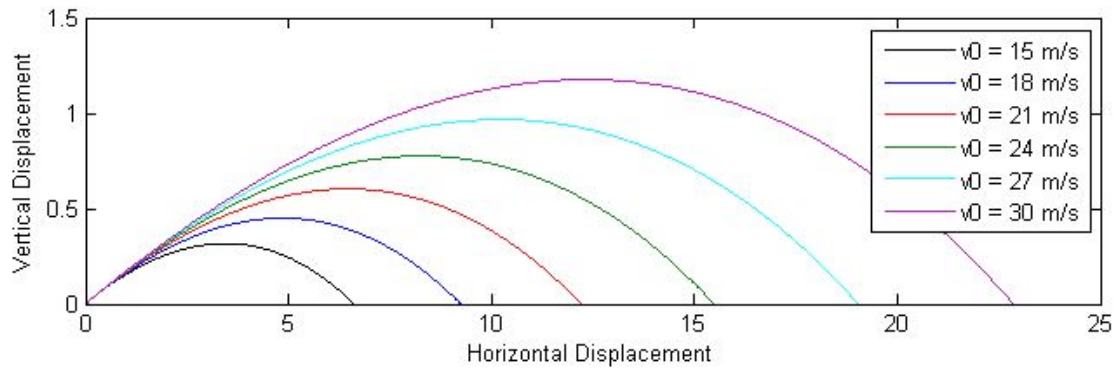


FIGURE 2. Ball trajectories for varying initial velocities (initial angle 10 degrees).

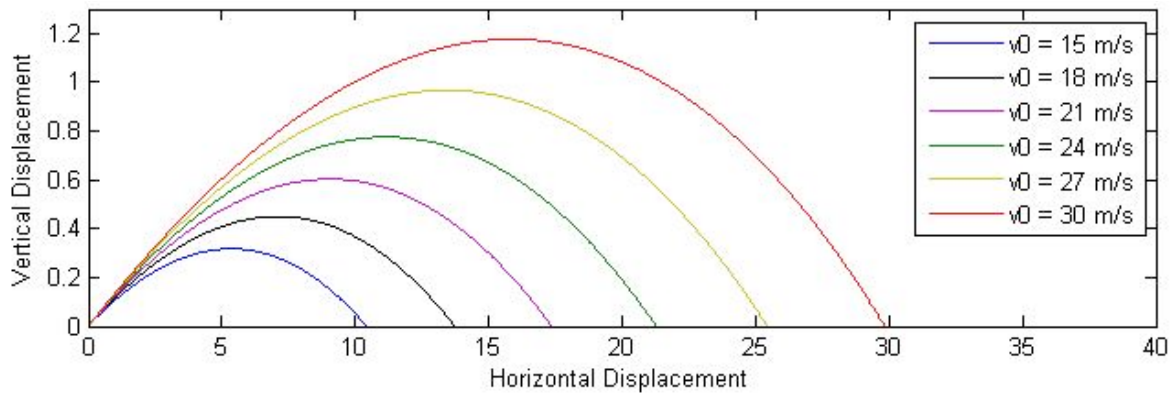


FIGURE 3. Ball trajectories for varying initial velocities with trailing wind 4 m/s (initial angle 10 degrees).

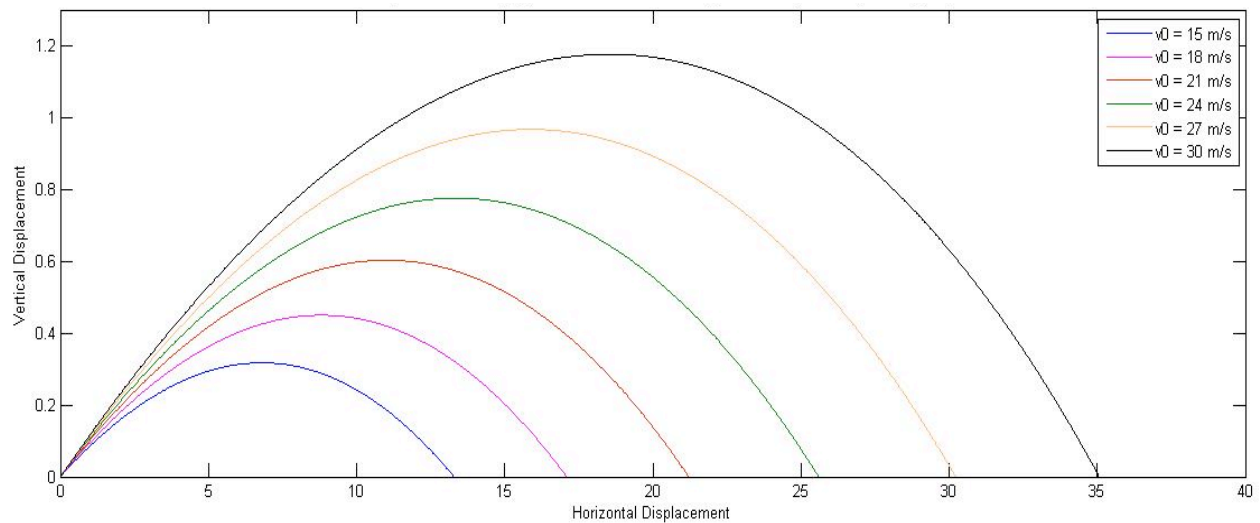


FIGURE 4. Ball trajectories for varying initial velocities with trailing wind 7 m/s (initial angle 10 degrees).

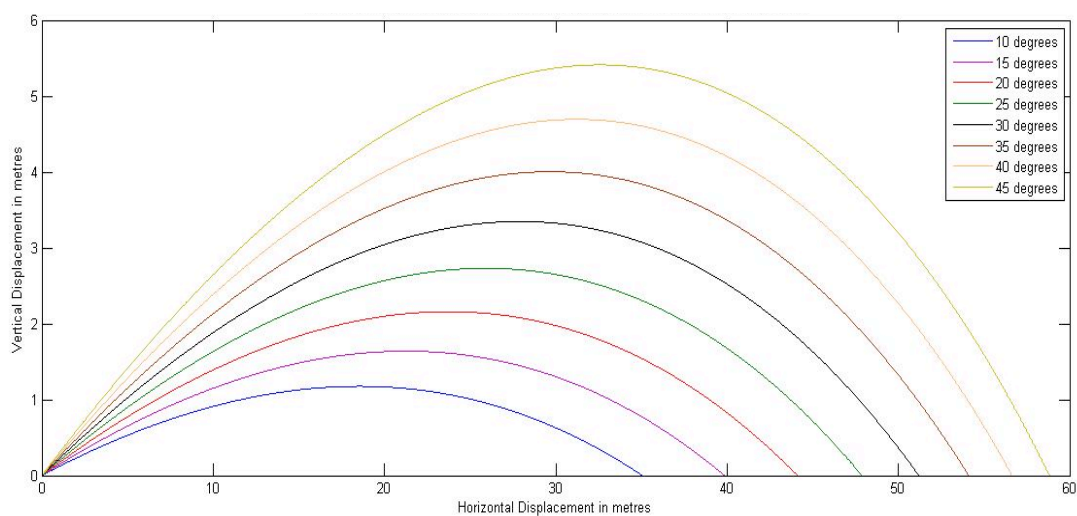


FIGURE 5. Ball trajectories for varying initial angles with trailing wind 7 m/s (initial velocity 30m/s).

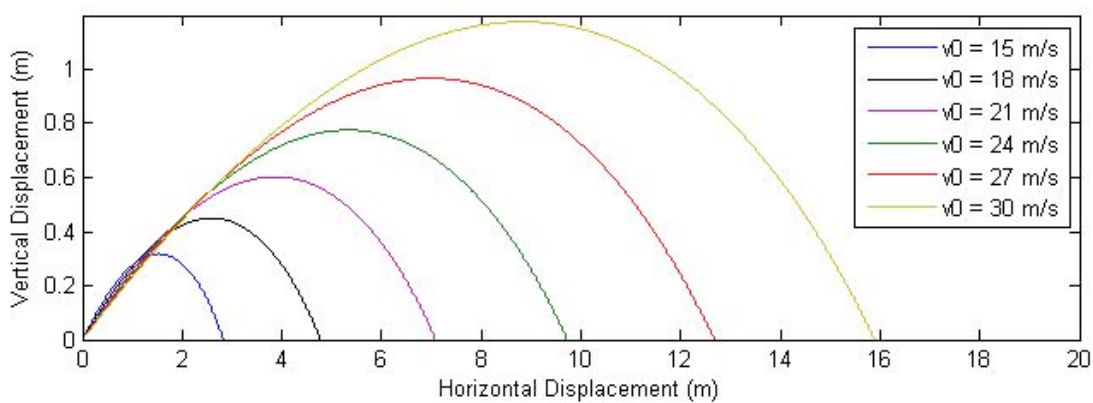


FIGURE 6. Ball trajectories for varying initial velocities with head wind 4 m/s (initial angle 10 degrees).

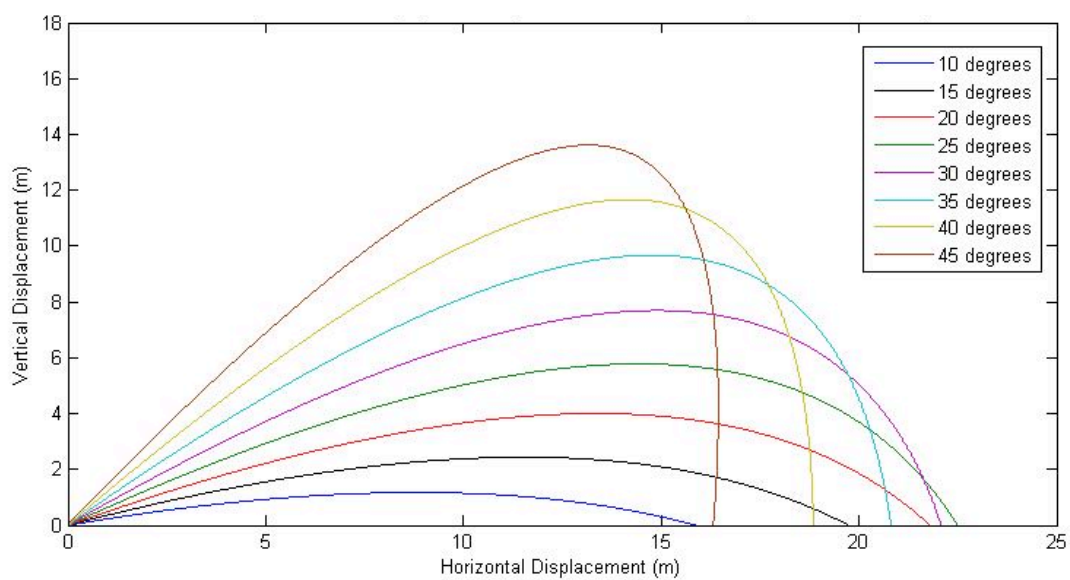


FIGURE 7. Ball trajectories for varying initial angles with head wind 4 m/s (initial velocity 30 m/s).

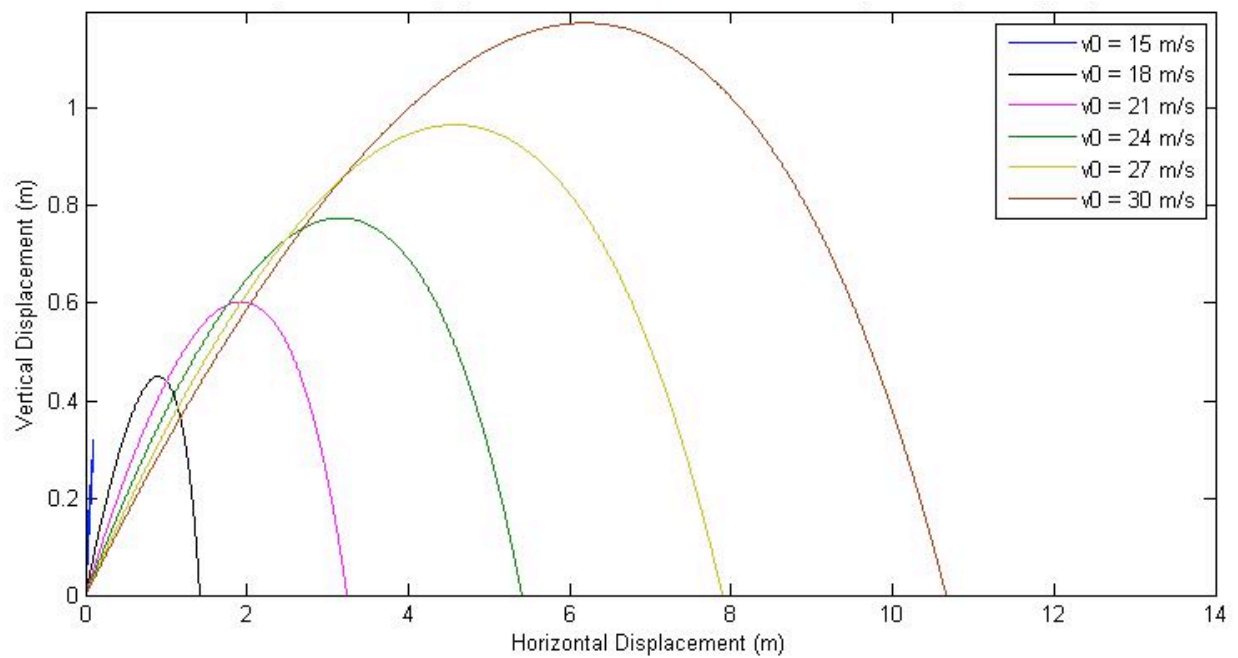


FIGURE 8. Ball trajectories for varying initial velocities with head wind 7 m/s (initial angle 10 degrees).

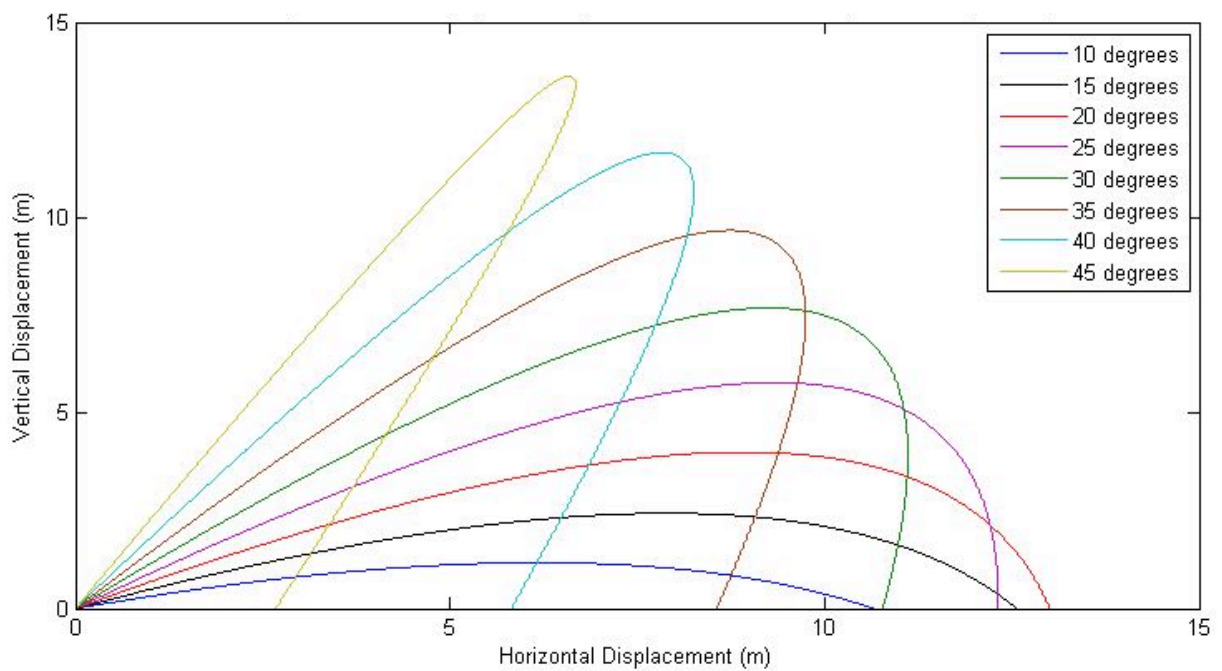


FIGURE 9. Ball trajectories for varying initial angles with head wind 7 m/s (initial velocity 30 m/s).

The following graphs show the vertical and horizontal displacements of the ball when back spin is applied. The amount of spin determines to amount of lift. First, we have the code.

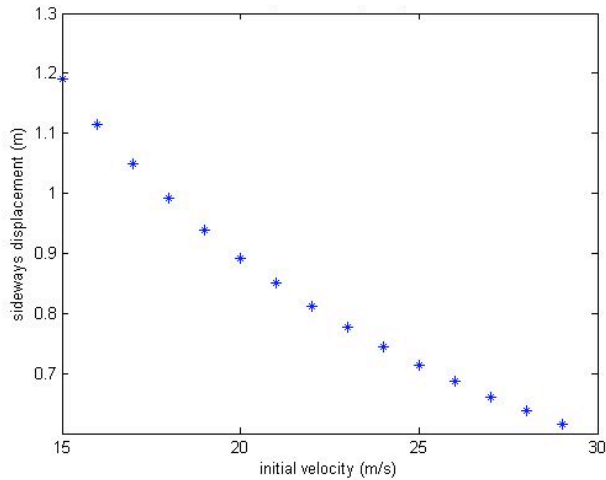


FIGURE 10. Initial velocity vs sideways displacement.

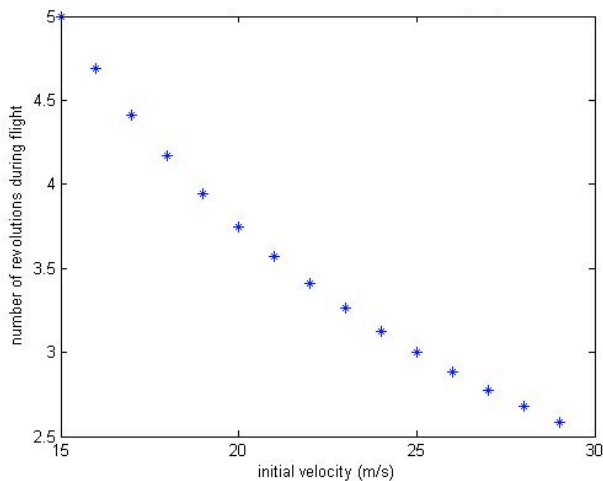


FIGURE 11. Initial velocity vs number of revolutions during flight.

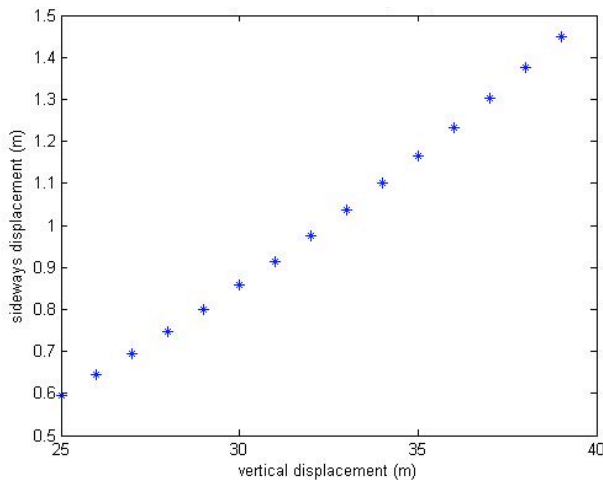


FIGURE 12. Vertical displacement vs sideways displacement.

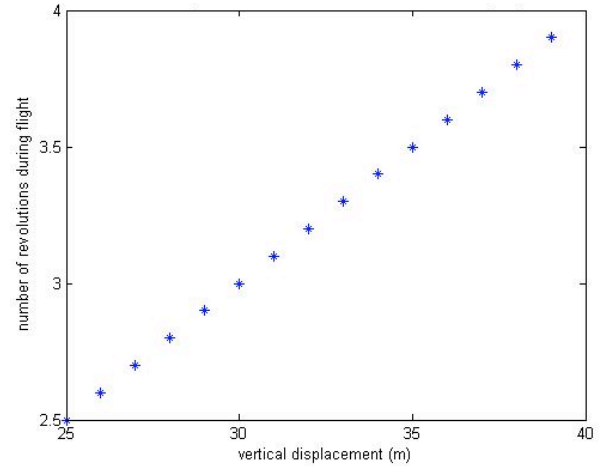


FIGURE 13. Vertical displacement vs number of revolutions during flight.

IV. DISCUSSION AND CONCLUSIONS

It is also important to note that the ratio of f/V appearing in equation (25) is related to the ratio of the rotational energy to the kinetic energy. This ratio is

$$\frac{E_R}{E_K} = \frac{\frac{1}{2} I \omega^2}{\frac{1}{2} m V^2}$$

and since:

$$I = \frac{2}{3} m a^2$$

$$\frac{E_R}{E_K} = 0.32 \left(\frac{f}{v} \right)^2, \quad V \text{ in } m s^{-1}.$$

For the example, a ball travelling at 30 mph ($13.4 m s^{-1}$) with a spin of 3 revolutions per second has a rotational energy of 1.6% of its kinetic energy. Our results clearly demonstrate the close link between our numerical simulation and reality. The trajectories with drag force acting against the motion of the ball will definitely affect its range as seen in figure 2 and 3. It clearly demonstrates that the maximum projectile range will not be achieved with a launch angle of 45° . The results show that a trailing wind will increase the range of the trajectory for the same velocity and launch angle with just drag whereas, a headwind will decrease the trajectory. This is quite consistent with the laws of Physics.

The results obtained when spin is applied shows that the ball bends perpendicular to the spin axis. If the spin axis is perpendicular to the ground the ball deviates left or right depending on if the rotation is clockwise or anti-clockwise.

If the spin axis is parallel to the ground, the spin has two forms. These are known as topspin where the ball is rotated towards the ground or lift where the rotation is towards the

air. The graphs show the more intense the upward rotation, the more lift is generated. This in turn has the effect of increasing the range of the trajectory. Soccer players utilize this technique in long range passing. These results can inform coaches and other personnel in the sport how to implement certain approaches in their training regime to develop greater accuracy and efficiency in medium and long range passing. A possible follow up paper can explore the how the materials, design, size and shape used in various soccer balls can influence their behaviour in flight.

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