A further look at capacitors in complex arrangements

Kenneth V. Cartwright¹, Patrick Russell¹ and Edit J. Kaminsky²

¹School of Mathematics, Physics, and Technology, College of The Bahamas, P.O. Box N4912, Nassau, Bahamas.
²Department of Electrical Engineering, EN 846 Lakefront Campus, University of New Orleans, New Orleans, LA 70148, USA.

E-mail: kvcartwright@yahoo.com

(Received 1 July 2014, accepted 26 November 2014)

Abstract

Recently, a previous contributor showed how the delta-wye conversion may be applied to find the equivalent capacitance of five capacitors arranged in a bridge configuration. In fact, that contributor gave an algorithm for this purpose which showed that the equivalent capacitance depends upon all five capacitors. However, in this paper, we point out that there is a special case where the equivalent capacitance does not depend upon one of the five capacitors.

Keywords: Combination of capacitors, Bridge circuits, Delta-wye and wye-delta conversions.

I. INTRODUCTION

In [1], Rizhov investigated the problem of finding the equivalent capacitance \( C_0 \) (with respect to terminals \( T_3 \) and \( T_4 \)) of five capacitors connected in a bridge configuration as shown in Figure 1.

![Figure 1](image-url)

His approach was to arrive at a system of seven unknowns, which were solved for one particular case. However, as pointed out by Atkin [2], this method “did not reveal how a general solution could be obtained, giving \( C_0 \) as a function of the five capacitances”. On the other hand, Atkin [2] showed how the delta-wye conversion may be applied for this very purpose. (Incidentally, the use of delta-wye and wye-delta conversions to solve problems of this sort is a well-known technique described in many electrical circuit analysis textbooks [3, 4]).

Unfortunately, the algorithm given by Atkin in [2] hides the fact that \( C_0 \) is independent of the value of \( C_5 \) if:

\[
\frac{C_3}{C_1} = \frac{C_4}{C_2} \Rightarrow C_0 = C.
\]

In fact, there is no hint at all in [1] or [2] that the equivalent capacitance might not depend upon \( C_5 \) It is the purpose of this paper to demonstrate this fact in two ways: by using i the algorithm given by Atkin [2] and ii the concept of a balanced bridge circuit.

II. REVIEW OF THE METHOD DESCRIBED BY ATKIN

In Atkin’s method [2], the delta configuration of \( C_1, C_2 \) and \( C_3 \) in Figure 1 is first converted to a wye circuit, given by \( C_5, C_7 \) and \( C_8 \) of Figure 2i, where:

\[
\frac{C_3}{C_1} = \frac{C_4}{C_2} \Rightarrow C_0 = C.
\]
III. DETERMINING THE EQUIVALENT CAPACITANCE FOR THE SPECIAL CASE

WHERE \( \frac{C_3}{C_1} = \frac{C_4}{C_2} \)

A. Using the algorithm given by Atkin

From Equations 2a and 1a:

\[
C_9 = \frac{C_3}{\frac{1}{C_3} + C_6} = \frac{C_3}{C_6 + \frac{C_3}{C_6} + \frac{C_5}{C_1}}
\]

Hence:

\[
C_0 = \frac{C_8 C_{11}}{C_8 + C_{11}}.
\]

Similarly, from Equations 2b and 1b:

\[
C_{10} = \frac{C_4}{\frac{1}{C_4} + C_7} = \frac{C_4}{C_7 + \frac{C_4}{C_7} + \frac{C_5}{C_2}}
\]

where \( K = \left( \frac{C_4 + C_2}{C_7} \right) C_5 + 1 \), and \( C = \frac{C_4}{C_1} \).

FIGURE 2. Simplification of Figure 1 using a delta-wye transformation.

Comparing Figure 2i with Figure 2ii shows that \( C_9 \) is the series combination of \( C_3 \) and \( C_6 \) and \( C_{10} \) is the series combination of \( C_7 \) and \( C_4 \). Hence:

\[
C_9 = \frac{C_3 C_6}{C_3 + C_6}, \quad (2a)
\]

and:

\[
C_{10} = \frac{C_4 C_7}{C_4 + C_7}. \quad (2b)
\]

Furthermore, comparing Figure 2ii with Figure 2iii shows that \( C_{11} \) is the parallel combination of \( C_9 \) and \( C_{10} \). Hence:

\[
C_{11} = C_9 + C_{10}. \quad (3)
\]

Finally, from Figure 2iii, it is clear that \( C_0 \) is the series combination of \( C_8 \) and \( C_{11} \).
With:

\[ C = \frac{C_4}{C_2}. \]

Substituting Equations 5 and 6 into Equation 3 gives:

\[ C_{11} = \frac{K}{K+C}(C_3+C_4). \]  (7)

Furthermore, Equation (1c) can be written as:

\[ C_8 = \left( \frac{C_1+C_2}{C_1C_2} \right) C_5 + 1 \frac{C_1C_2}{C_5}, \]

\[ = \frac{K}{K+C} \frac{C_1C_2}{C_5}. \]  (8)

Substituting Equation (7) and Equation (8) into Equation (4) gives:

\[ C_0 = \frac{K^2 \frac{C_1C_2}{C_5} + 1}{K+\frac{C_1C_2}{C_5} + \frac{K}{K+C}(C_3+C_4)} \]

\[ = \frac{K}{K+C} \frac{C_1C_2}{C_5} \left( \frac{K(C_3+C_4)}{K+C(C_3+C_4) + \frac{C_1C_2}{C_5}} \right), \]  (9)

Further simplification of Equation (9) gives:

\[ C_0 = \frac{K(C_3+C_4)}{C_5 + K+\frac{C_1C_2}{C_5}} \]

\[ = \frac{C_1C_2}{C_5} \frac{K(C_3+C_4)}{K+C(C_3+C_4) - \frac{C_1C_2}{C_5}}, \]

\[ = \frac{K(C_1C_4)}{1+C(C_3+C_4) + \frac{C_1C_2}{C_5} \frac{C_5}{C_1C_2}} \]

\[ = \frac{K(C_1C_4)}{1+C(C_3+C_4) + \frac{C_1C_2}{C_5} \left( \frac{C_5}{C_1C_2} \right) \frac{C_5}{C_1C_2}} \]

\[ = \frac{K(C_1C_4)}{(1+C) \left[ 1 + \left( \frac{C_1+C_2+C_3+C_4}{C_1C_2} \right) \frac{C_5}{C_1C_2} \right]} \]

\[ = \frac{K(C_1C_4)}{(1+C) \left[ 1 + \left( \frac{C_1(1+C)+C_2(1+C)}{C_1C_2} \right) \frac{C_5}{C_1C_2} \right]} \]

Hence:

\[ C_0 = \frac{K(C_3+C_4)}{(1+C)K}, \]

\[ = \frac{C_3}{1+C} + \frac{C_4}{1+C}, \]

\[ = \frac{C_3}{1+C} + \frac{C_4}{1+C}, \]

\[ = \frac{C_5}{C_1} + \frac{C_4}{C_2}, \]

\[ = \frac{C_2C_1}{C_1+C_3} + \frac{C_4}{C_2}. \]  (11)

As can be seen from Equation (11), the equivalent capacitance does not depend upon \( C_5 \) as claimed. In fact, the value given by Equation (11) is the same value that would be given if \( C_5 \) is not connected to the circuit. Hence, if \( \frac{C_3}{C_1} = \frac{C_4}{C_2} \), the bridge circuit behaves the same regardless of the value of \( C_5 \).

**B. Using the concept of the balanced bridge**

As can be seen from the above, it is straightforward but tedious to derive the equivalent capacitance for this special case. Fortunately, there is a well-known alternative way of deriving this same result using the idea of a balanced bridge \[3, 4\]. For completeness, we will review this method very briefly.

Consider the bridge circuit of Figure 3. If the current through \( Z_5 \) is zero, the bridge is said to be balanced and \( Z_5 \) will have no effect on the bridge circuit. In fact, \( Z_5 \) can now be an open or short circuit.

If we replace \( Z_5 \) with an open circuit, \( i.e. \), remove \( Z_5 \) the total impedance will be given by the parallel combination of \( Z_1 + Z_3 \) and \( Z_2 + Z_4 \) giving rise to Equation 11.

Furthermore, to ensure that the bridge is balanced, it is easily shown that \( \frac{Z_1}{Z_3} = \frac{Z_2}{Z_4} \), or equivalently, \( \frac{C_1}{C_2} = \frac{C_3}{C_4} \).
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To see this, note that the current through $Z_5$ is zero when the voltage across $Z_5$ is zero, \(i.e.:\)

$$V_5 = V_3 - V_4 = 0.$$  \(\text{(12)}\)

Furthermore, when \(I_5 = 0\), we apply the voltage divider rule to obtain:

$$V_5 = \frac{Z_1}{Z_1 + Z_3} E = \frac{1}{1 + Z_1/Z_3} E,$$

and:

$$V_4 = \frac{Z_4}{Z_2 + Z_4} E = \frac{1}{1 + Z_4/Z_2} E.$$  \(\text{(13)}\)

Substituting Equation 13 and 14 into Equation 12 gives the desired result:

$$\frac{Z_1}{Z_3} = \frac{Z_2}{Z_4}.$$  \(\text{(14)}\)

IV. CONCLUSIONS

We have shown that the equivalent capacitance of five capacitors connected in a bridge configuration does not depend upon one of the capacitors for a special case. We showed this by using the algorithm provided by Atkin and also by using the concept of a balanced bridge.

REFERENCES