Implementation of a didactic proposal on fundamental concepts of quantum mechanics with students of a professional master's degree in physics teaching

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Abstract

This paper presents the findings of a research on the acquisition of the concepts of *physical system*, *dynamical variables*, *state of a physical system*, and *time evolution* in Quantum Mechanics, based on the learning theories of David Ausubel and Gérard Vergnaud. This investigation involved six students in a Professional Master's Degree in Physics Teaching, and it consisted of the implementation of a didactic proposal that comprised a 12-hour-long expository-dialogued presential course, which also included extra-classroom tasks and interviews. Evidences of the occurrence of meaningful learning of the course contents were obtained, both at the operative and at predicative forms of knowledge, with some limitations in the assimilation of some of these concepts. We noticed a few difficulties in the assimilation of the concepts of *physical system, dynamical variables,* and higher degree of difficulty in the assimilation of the concept *state of a physical system*. We attained a relevant point, which was the modification of the concept *time evolution* that had been already present in the students' cognitive structure, though in a nonspecific way before instruction.

Keywords: meaningful learning; quantum mechanics; physics teaching.

Resumen

Este trabajo presenta los resultados de una investigación sobre la adquisición de los conceptos del *sistema de física, variables dinámicas, estado de un sistema físico,* y *el tiempo de evolución* en la Mecánica Cuántica, basado en las teorías de aprendizaje de David Ausubel y Gérard Vergnaud. En esta investigación participaron seis estudiantes de un Grado Profesional de Maestría en Enseñanza de la Física, y consistió en la implementación de una propuesta didáctica que comprendía 12-horas-de duración curso presencial expositivo-dialogado, el cual también incluyó clases-extra tareas y entrevistas. Las evidencias de la ocurrencia del aprendizaje significativo de los contenidos del curso que fueron obtenidos, tanto en la operativa y en las formas de predicación del conocimiento, con algunas limitaciones en la asimilación de algunos de estos conceptos. Nos dimos cuenta de algunas dificultades en la asimilación de los conceptos de Física del *sistema físico, variables dinámicas,* y un mayor grado de dificultad en la asimilación del concepto *estado de un sistema físico*. Hemos alcanzado un punto relevante, que fue la modificación del concepto *evolución del tiempo* que ya había estado presente en la estructura cognitiva de los estudiantes, aunque de una manera no específica antes de la instrucción.

Palabras clave: Aprendizaje significativo, mecánica cuántica, enseñanza de la física.

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I. INTRODUCTION

One of the current problems in the teaching of physics, as pointed out by Olsen [1], is that the physics curriculum at high school is largely outdated in relation to the knowledge that has been constructed in this field. The researchers highlight that although it has been over a century since the introduction of Planck's constant in physics, school curriculum quite often remains strongly influenced by what has been labeled the Classical Physics (CP) that is, Classical Mechanics, Classical Electromagnetic Theory, and Classical Thermodynamics.

In the specific case of investigations on the teaching of Quantum Mechanics (QM), researchers draw attention to the fact that even though many improvements have occurred in the area, until the last decade of the 20th century, reports of studies that aimed at students' conceptions as well as the implementation of didactic proposals in the literature [2] were quite scarce. Despite the tendency of growth in this area (ibid), more studies are necessary to provide a clearer perception of what prior knowledge students already have and how they assimilate new contents in this field of knowledge. Çaliskan *et al*. [3] stress, for example, that the number of research papers on students' understanding of quantum mechanics is lower than those on classical physics.

In addition to the lack of research in this area and the existence of an outdated curriculum, hybrid models, which derive from the students' semi-classical conceptions, can also hinder the teaching of quantum mechanics [4, 5]. The few existing studies on the students' conceptions seem to point out to the construction of models that have concomitantly quantum (for example the quantization of energy) and classical attributes (such as, the existence of defined trajectories). Tsarpalis and Papaphotis [5] distinguish hybrid models from the Old Quantum Theory models (a period roughly starting in 1900 with the presentation of Planck's hypothesis up to 1926, with the first articles about the formulations of Quantum Mechanics), considering that these OQT models are not alternative conceptions, inasmuch as these authors classify them as a setback to the learning of quantum mechanics.

Another aspect related to the difficulties already mentioned is the use, sometimes in excess, of a mathematical approach to the kind of teaching that usually happens at introductory levels. That is, it stresses the operator-mathematical aspect more than the conceptual one. The lack of a conceptual bias[1] in those teaching processes might be responsible for the students' development of hybrid models, because quite often the students cannot identify and distinguish the concepts pertaining to the domains of quantum physics and classical physics.

This paper is based on the ideas presented above, and it aims at reconciling similarities and at emphasizing the differences between features of quantum physics and classical physics through the use of a fundamentally conceptual approach. Our theoretical framework is grounded on the idea of David Ausubel [6] that the use of the principles of progressive differentiation and integrative reconciliation can facilitate meaningful learning of concepts in a given field of knowledge, which allows for a delay in the use of mathematical tools until students have acquired/constructed the conceptual knowledge they need. In other words, our prime concern is the semantic knowledge over the syntactic one, that is, we emphasize the meaning throughout the teaching process, so that, when mathematical equations are presented, they tend to express the conceptual generality of the presented propositions.

The present study was developed in the second half of 2010 and it involved six students of the Professional Master's Degree in Physics Teaching at the Federal University of Rio Grande do Sul, Brazil. It should be mentioned here that the aim of this course is the qualification at graduate level of inservice high school teachers, endeavoring to attain a positive influence in their profession as teachers and in their classroom activities.

II. THEORETICAL FRAMEWORK

Gérard Vergnaud [7] considers and analyses two types of knowledge developed by human beings, that is, *the predicative form of knowledge*, explicit and formalized, and *the operative form of knowledge*, implicit in action and not necessarily formalized. It is important to highlight that, just as Vergnaud does, both forms of knowledge carry meaning, but in distinct ways.

Knowledge in the predicative form, as it is explicit, can be discussed, whereas the operative knowledge, as it is implicit in action (which, in the case of physics teaching, is strongly associated to problem solving), is seldom analytical. As meaning is a fundamental part of the types of knowledge Vergnaud contends, then, that conceptualization is the cornerstone to cognition [7].

For this reason, we adopted not only David Ausubel's theory, that emphasizes the learning of explicit and articulated knowledge, but also Vergnaud's theory that considers the development of knowledge in the operative form.

Next, we present the fundamental ideas of David Ausubel's theory, that will allow us to explain the general processes of knowledge acquisition in the predicative form, which will be followed by a synthesis of the fundamental concepts of Gerard Vergnaud's Conceptual Fields Theory.

A. Ausubel's meaningful learning theory

As the discussion about Ausubel's Meaningful Learning Theory in the literature is very broad, we have just focused on those aspects we have considered fundamental for both the construction of the didactical unit and for the analysis of the obtained data in the research process.

Meaningful learning, according to Ausubel, is a process of acquisition of new knowledge so as to generate new meanings. Meaning, as Ausubel proposes, is a differentiated, clear, and conscious entity.

"... meaning is not an implicit response, but rather a clearly articulated and precisely differentiated conscious experience that emerges when potentially meaningful signs, symbols, concepts, or propositions are related to and incorporated within relevant components of a given individual's cognitive structure on a nonarbitrary and nonverbatim basis" [6]

This process is divided into three stages that he calls acquisition (the cognitive processing of information derived from external environment), retention (maintenance of information in the knowledge structure), and obliteration (residual forgetting of information). Obviously, the process occurs distinctively in each person, and Ausubel takes into

[1] This emphasis on concepts should be understood as a means to facilitate the construction of mental models that are compatible with conceptual models scientifically accepted. For further details, see Greca (2000).

account two crucial factors to highlight the differences in learning verified in different individuals: prior knowledge and the predisposition to learn meaningfully.

It is relatively clear the role prior knowledge plays in learning, because the more elaborated, stable and differentiated it is in one's cognitive structure, the easier it is its relation to the information to be learned meaningfully, *i.e.*, the higher the probability an individual has to retain for a longer period of time the acquired information.

The role of predisposition to meaningful learning is also a remarkable matter. Some people are more willing to relate information from the external world to their cognitive structure in a non-arbitrary and non-literal way, for example, students who are able to develop their own ideas in a natural way, either by presenting examples or by building a coherent discourse. There are other students who, for a variety of reasons (unwillingness to learn meaningfully, lack of relevant previous information, etc.) tend to arbitrarily store new information in their cognitive structure, often expressed by the fact that the relationship between, or among, concepts is highly peripheral, or deprived of enough evidence to support it, and it can be literal, substantively non-replicable (with the student's own words).

These two extremes of a continuum, Ausubel denominates meaningful learning (non-arbitrary and substantive) and rote learning (arbitrary and verbatim). It seems important to stress that the situation in which a student is either an exclusively rote or exclusively meaningful learner will hardly happen, since even the task of learning an equation, for example, carries rote learning elements [2] in it, notwithstanding that it is a potentially meaningful task.

In physics teaching, in general, and in the teaching of quantum mechanics, in particular, concepts and propositions play a fundamental role, for they constitute an essential part of scientific language. Concepts express regularities in events and objects, and they are endowed with characteristic attributes, whereas propositions are compounds of words that contain concepts. Besides meanings, concepts usually have names that facilitate their retention, since they provide a very important language economy that favors the reproduction of ideas present in the students' cognitive structure. Propositions carry both the semantic and the syntactic aspects of language, and are crucial to the construction of knowledge in the classroom.

It seems worth going back to the idea that there are differences in terms of prior knowledge that are present in people's cognitive structure. We will call subsumer this knowledge an individual already has in his/her cognitive structure (since prior knowledge is responsible for the subsumption of new ideas). Subsumers have distinct levels of clarity (depending on their level of definition and accuracy), differentiation (according to how differentiated they can be between and/or among themselves), stability (their capacity of not being modified), and availability (according to how available they are for a given learning task).

Let us analyze, for instance, the concepts of heat and temperature a given individual has, and whose conceptions have been supposedly externalized, as a didactic exemplification. He/she might understand the concept of heat as a substance that increases the temperature and he/she might understand the concept of temperature as something similar to the thermal sensation of hot and cold. For him/her, the concepts present a good level of clarity, since they can be easily defined, are differentiated, since he/she can differentiate them from each other and/or among themselves, and these concepts are already available for a learning task, though they do not have a necessary coherence according to those of the scientific community. Then, what will differentiate the product of learning is the stability of the concept. Considering that these concepts are so deeply rooted in the cognitive structure of some students it is possible to understand the reason why they are so difficult to change. This difficulty of conceptual change is directly linked to the stability of a given concept or proposition.

These cognitive structure variables have been fundamental for studying how learning has been developed along the didactical intervention we describe here. They shed light on what concepts students have shown more difficulty in learning.

Ausubel, in order to deal with these cognitive structure variables, suggests four programmatic principles (progressive differentiation, integrative reconciliation, sequential organization, and consolidation), out of which we will analyze two of them in a more in-depth way, that is, progressive differentiation and integrative reconciliation.

Progressive differentiation implies that we should first present the more general, inclusive, and abstract [3] concepts, and then we should depart from these to get to the more specific ones. Ausubel argues that the natural order of learning in the human cognitive apparatus goes from the more general to the more specific and, so, we can justify this principle.

The second principle, integrative reconciliation, emerges because, if we just follow the progressive differentiation principle, potentially relatable concepts would be indefinitely differentiated and, thus, the aim of constructing a cognitive structure (made of conceptual, propositional, and/or representational linkages) would be flawed. So that we would have many disjointed cognitive branches derived from learning in the classroom. In other words, we would not find linkages between/among the learned ideas. This principle has to be intercalated with the

[2] Although representational learning incorporates elements of meaningful learning, it seems closer to the rote learning end of the continuum [1]. Ausubel, when talking about representational learning, which is needed for learning equations, demystifies the myth of the existence of a student as a totally meaningful learner.

[3] An abstract concept is understood here as a synonym of a general concept, that is, a concept that has smallest possible number of concrete criterial attributes (Ausubel, 1980, p.80).

progressive differentiation principle if we want the teaching process to achieve better results in both the structural aspect of knowledge and in knowledge recall, which is essential to a higher level of retention related to the learning process.

Ausubel, then, presents what can be called a good theory, which does contend with the acquisition of knowledge in its operative form, but that seems to suit rather well the explanation of knowledge learning processes in the predicative form (explicit and articulated), for it takes into account the cognitive structure and meaning as organized and explicit entities. A theory that can potentially complement Ausubel's Theory in its explanation of learning processes and in the development of cognitive structures is Gerard Vergnaud's Conceptual Fields Theory [7].

B. Vergnaud's conceptual field theory

Vergnaud considers that most of the knowledge and competencies we develop, improve, differentiate, and deteriorates throughout our lifetime, and they depend on the situations with which we have faced. Vergnaud uses Piaget's concept of scheme to take into account this characteristic.

Scheme, in Vergnaud's words, consists of the invariant organization of behavior when the individual faces a class of situations. Those schemes are dynamic functional totalities that are not restricted to sensory-motor activities, but that also include the intellectual activity [7]. The situations, to which Vergnaud refers, have a dual nature (they are connected to both motor and cognitive aspects), so that taking hold of an object or solving a linear algebra problem can constitute actions involved in situations for which we need a scheme if we want to master them.

A scheme is not an invariant behavior, and not at all a stereotype, but it is an organization of behavior constructed in an invariant form. It is the unit of analysis of knowledge in the operative form and it is composed of items such as:

- *goals and anticipations;*
- *action rules, of provision and control of information;*
- *possibilities of inference;*
- *operational invariants.*

By *goals* and *anticipations*, Vergnaud considers that a scheme addresses to a class of situations in which one can describe the purpose of the activity, or can expect some effects or phenomena [7]. This is associated to the prediction for a solution of a given problem. An expert can, for example, anticipate the coherent solution of a problem on gravitation by using the scientific knowledge he/she has, whereas a novice might anticipate an incorrect solution to the problem. Both are anticipations, although in the context of physics, the first one is considered valid, whereas the second one is not.

Action rules allow for the generation of continuity of actions of transformation of reality, of information provisioning, and of action controls and results, providing for the guarantee of success of an activity in a context of permanent evolution [7]. A physics student, while solving a problem in this field of knowledge, can get to a point in which he/she must make a choice of one among the several paths to the continuation of the problem solving. He/she might use an action rule such as, for example, "calculate the kinetic energy and add it up to potential energy" or "decompose down the forces in a tangential component and in a radial one", or even the use of an incorrect rule of action, such as, "add the centripetal force to all external forces perpendicular to the movement direction, and equal it to the resultant force". In other words, it is a mechanism of the type "do this" and/or "store that".

The scheme brings within it *possibilities of inference*, because any activity requires calculation of the "here and now" type, in situations or inferences like "*if we have x, y will occur*" [7], which are conditional rules. A good example of this is when a more "conscious student" performs the calculus of an electric field derived from a sphere and comes up with a result like the one due to a spheric electric charge, and concludes that the shape of the electric field is associated to the symmetry of the problem, that is similar for punctual charges.

Operational invariants are entities that constitute the implicit or explicit conceptual base of the schemes, that is, the specific content. The operational invariants are divided into *concepts-in-action* and *theorems-in-action* [7]. These categories form the base of conceptualization of the scheme. For Vergnaud,

"A theorem-in-action is a proposition considered to be true about the real; a concept-in-action is a category of thought considered as pertinent/ relevant" [8].

This cognitive psychologist also relates these things to the schemes,

"The main theoretical aim of the scheme is to provide the indispensable linkage between behavior and representation. On the other hand, it is the operational invariants that form the essential articulation, since perception and the search and selection of information are entirely based on the system of concepts-in-action available in the individual (objects, attributes, relationships, conditions, and circumstances) and on the theorem-in-action underlying his/her behavior" [8].

It is necessary to distinguish, even more, the two concepts. The schemes need concepts, which are fundamental entities to categorization. So, we have to select a small portion of the available information, and for this selection to happen, we must be able to classify "things", that is, we must be able to group objects, predicates, conditions that can or cannot be relevant in the domain of a given situation [7].

Theorems are, on the other hand, false or true. Theorems-in-action are propositions that group up concepts and these propositions are considered to be true about reality, which is what we consider to be correct. The existence of these theorems allows us to reason about inferences and event anticipations [7].

Let us apply these given examples to action rules to clarify further the concept of operational invariants. Kinetic energy, potential energy, force, tangential component are concepts-in-action, whereas the propositions held as being true about the reality, invariants to a given class of situations that are made up from the concepts-in-action, configure themselves as theorems-in-action, as for example, the assumption (erroneous) of the centripetal force as a force to be included the system in circular motion.

Scheme is Vergnaud's essential unit of analysis to study the subject-in-situation, and it stands as the crucial element for the conceptualization of the real (the cognitive development nucleus), because it is embedded in the situations. Thus, Vergnaud claims that it makes more sense to deal with the interaction between scheme and situation instead of the interaction between subject and object. For Vergnaud, then,

"... education and instruction have to contribute to build a diversified repertoire of schemes avoiding, moreover, that these schemes become passive stereotypes" [7].

Vergnaud has built an explicative conceptual structure to the acquisition of knowledge, from the point of view of interaction between scheme and situation. However, we have not mentioned yet what has been learned or mastered, nor how this knowledge has been represented. Therefore, it becomes fundamental to introduce the concept of Conceptual Field, which for Vergnaud means

"…large groups of situations whose analysis and handling require several types of concepts, procedures, and symbolic representations that are linked to each other" [9].

This definition can become even clearer when this author introduces the notion of concept, which for him constitutes a triplet of sets:

$$C= (S, I, R),$$

in which S is the set of situations that make the concept useful and meaningful; I is a set of operational invariants that an individual might use to handle situations; R stands for the set of representations (linguistic, graphic/pictoric or gestural) that can be used to represent the operational invariants, situations and procedures [10].

Symbolic representations (the only element of this triplet of sets we have not yet detailed), allow for the representation of concepts and theorems-in-action, as well as situations. Some representations are more powerful than others, but they cannot be handled with before the individual has incorporated their meaning to his/her cognitive structure. Graphics often seem more effective with some people, while others prefer the formalism of equations. However, ideally, a person should proceed dealing with the largest possible number of representations.

It is clear, then, the meaning of the proposition stating that an individual masters a given conceptual field through the interaction of his or her schemes with the situations that he or she faces. The scheme, fundamental mechanism of behavior analysis, is composed of concepts-in-action and theorems-in-action, which are the contents of the scheme, something that is not seen in Piaget's schemes, although Vergnaud considers the concept of schemes a great Piagetian heritage. The scheme is the departure point from which an individual can master the situations related to the conceptual field.

III. METHODOLOGY

In this section we will discuss the relevant details about the methodology used in this study. For this purpose, it seems beneficial to divide it into two parts: teaching methodology and research methodology.

A. Teaching Methodology

The didactic intervention was carried out with students of the Professional Master's Degree in the Teaching of Physics at Federal University of Rio Grande do Sul, Brazil, and it comprised 12 hours of required class attendance. The analyzed group was composed of six students, four of which had an undergraduate degree in Physics Teaching and the other two had a similar degree in Mathematics Education. In this group, three students had not had quantum mechanics in their undergraduate courses.

The content of this intervention included the concepts of physical system, dynamical variables, state of a physical system and time evolution. The first reason for choosing these topics was the Ausubelian premise of progressive differentiation since these concepts are some of the most general ones in quantum mechanics; therefore, they must be presented first, and then progressively differentiated (and integratively reconciled as well). Another reason for insisting on these concepts is related to the fact that the processes of teaching and learning concepts, such as state of a physical system and time evolution are hardly investigated in the literature. The few existing evidences point to few studies, such as the ones by Greca and Herscovitz [11], Rocha [12] and Singh [13].

The focal concept of our approach was *time evolution*, and its importance to the study of physics stands as the major reason for our emphasis. This concept brings about other very important concepts for the structuring of science, such as the concepts of *predictability* (the ability to anticipate) and *causality* (to know an effect from its causes). In addition, another relevant reason for this discussion, which is more focused on the state of research in physics teaching, is the small investment that is made in research on the learning and teaching processes on Time evolution in Quantum Mechanics. Thus, in our view, this concept plays a major role in research in physics teaching, in general, and in research in teaching quantum mechanics, in particular.

The concepts *state of a physical system* and *dynamical variables* have been considered more general than the concept of *time evolution*, because this one can be considered an attribute related to the other two concepts. Consequently, the concept of time evolution can be seen as a concept that possesses a higher degree of specificity. In order to study the process of temporal evolution, it is necessary to specify what is involved in this process. Since in quantum mechanics it can happen/occur both in the states and in the operators through the formulations of Schrödinger and of Heisenberg respectively, we insist on the differentiation of these concepts.

This approach has also considered another concept, *physical system*, which seems to be even more important than the three others we have previously mentioned. The concept of *physical system* is fundamental for the characterization of the object of study in physics and, hence, must be included in a proposal whose objective is to facilitate the meaningful acquisition of the concepts involved in it.

The intrinsic sequential organization of the concepts in the content was used, that is, *physical systems→ dynamical variables and state of a physical system→ time evolution*, which facilitated, in our opinion, the development of a process of progressive differentiation and, afterwards, of integrative reconciliation.

The teaching assumptions we adopted were also based on Gérard Vergnaud's fundamental idea according to which the progression of a given conceptual field occurs with an increase in the mastering of situations that constitute the field. Thus, we have basically adopted these three situations: the Stern-Gerlach experiment, the hydrogen atom, and the ammonia molecule. They have been used both to illustrate and facilitate the identification of the possible operational invariants held by the students.

The choice of the Stern-Gerlach experiment situation, besides its great importance to the context of QM, also presents pragmatic and epistemological reasons. The practical reason for this choice resides in the fact that the experiment is linked to the description of a two-level system, which is the simplest possible (mathematically speaking) among those rich enough to explore specific outcomes of systems in QM. The epistemological reason relates to Vergnaud's premise that one situation can ascribe meaning to several concepts. It seems to be the case of this experiment that exemplifies the involvement of the concepts of angular momentum, spin projection, compatible dynamic variables, incompatible dynamic variables, states of physical systems, superposition of states, measures of observables, constitution of atoms (silver, in this case), etc.

The *hydrogen atom* has also been selected because of its importance in QM, in addition to epistemological and pragmatic reasons. The pragmatic part of the selection lies in the students' prior knowledge about the structure of this quantum entity, which constitutes the simplest atomic system, that is, a proton and an electron (in a simple model that ignores the spin of particles and its effects) interacting via electromagnetic interaction. The epistemological part, similarly to the case of the Stern-Gerlach apparatus, is associated to whatever concepts this situation can make meaningful, such as, for example, uncertainty relations between position and momentum variables, eigenvalue problems, orbital angular momentum.

In relation to the *ammonia molecule*, the pragmatic bias of this selection is as important as the epistemological aspect. It is a system in which we might "see" the superposition of states, that is, it is feasible in a laboratory scale, and it is also helpful in the approach of the quantum-tunneling phenomenon. Moreover, it is favorable situation to the facilitation of changes of the states of the system through time evolution.

We must also emphasize the dialectic aspect between concepts and situations in content presentation. Firstly, we have dealt with the situations and afterwards with the concepts to be differentiated (progressively) and reconciled (integratively).

The students' prior knowledge has been taken for granted as that of some classical processes of time evolution, physical systems, and dynamical variables. This was confirmed with the pre-test taken in the first class. The initial evaluation showed a high level of generality of the already mentioned concepts, while the first situations suggested a reasonable confidence of the students in handling classical physics (more evident in mechanics) when compared to quantum physics.

The didactic resources developed for this course implementation can be found in Pantoja [14]. The classes, based on the referred didactic material, were divided in such a way that the concept of *physical system* could be handled at the first day of class and the concept of *dynamical variables* could be approached at the second day of class, associating it with the one of physical system. For classes three and four, we scheduled a discussion about the concept of *state of a physical system*. The intervention came to an end with two meetings for the discussion of the concept of time evolution. The strategy used in class consisted of the presentation of a problem-situation to the students so that they could express their understanding of the concepts to be discussed in class and, after contextualizing the content, to start developing the processes of progressive differentiation and integrative reconciliation.

B. Research Methodology

Research procedures applied here are associated with qualitative inquiry that has been quite closely related to the so-called naturalist paradigm, according to which reality is considered as possessing features that can evidence multiple forms. The fact that meaning is considered, in this theoretical framework, as an idiosyncratic entity is strongly related to the assumption that reality is something individually grasped, though with many shared aspects in a subject matter, such as, for instance, the misconception students develop, or those concepts experts use in a rather similar way.

The fact that meaning has crucial relevance in qualitative research deserves a more detailed explanation. Strictness applied to study how meanings are manipulated is linked to the use of triangulation techniques for research. This process reduces the natural bias of qualitative research. Therefore, in this study, multifaceted data collection, which we have developed as instances of formative evaluation, constitutes the resource used to add more credibility to it.

Data collection was made through paper-and-pencil problems (six sets in total—one for each two-hour class period), construction of concept maps, and a semi-structured interview, together with field notes that expressed teacher-researcher's impressions about knowledge acquisition by the group of students.

Furthermore, for ethical reasons, the participants' names are kept undisclosed as they are identified with a different number for each one of them.

The last point to discuss in relation to methodology is how the data analysis has been done, before going to research findings. We used two distinct procedures to study the students' acquired knowledge, which were both guided by content analysis [15], that is, an analysis of predicative knowledge and one of operative knowledge.

Analysis of predicative knowledge involved the investigation of meaningful assimilation patterns, that is, the study of the product of non-verbatim and non-literal interactions between relevant prior knowledge in the cognitive structure with knowledge that was approached along the intervention, based on the content verbally expressed by the students. Our analysis is mostly founded on the changes in the subsumers in the students' cognitive structure: changes in the cognitive structure variables and /or in the assimilation of new criterial attributes to the early existing concepts.

The analysis of the operative knowledge consisted of an investigation of concepts-in-action and theorems-in-action that students implicitly used in the mastery of a set of situations. This quest implied a search of theorems and concepts that were used similarly in distinct, though correlated situations. The attempt at reconstituting the thinking operations that students had used was grounded on the inferences derived from an indicator, such as, for example, a symbolic representation used in an apparently illogical way or in lapses demonstrating lack of coherence in the answers. This type of analysis has been fundamental in order to complement the analysis of predicative knowledge.

This information has allowed us to reach the findings of this study.

IV. FINDINGS

Discussion on the assimilation patterns evidenced by students, in relation to some concepts and ideas are treated here with the label *predicative knowledge analysis*, whereas the handling and presentation of possible operational invariants appear in this article in the subsection operative knowledge analysis.

As we have already mentioned, we identify the students with a number (student 1, student 2 and so on); thus, each student has been assigned a number from one to six.

A. Predicative Knowledge Analysis

Considering the whole group of subjects involved in the research, three of them have displayed, prior to instruction, concepts at a general level; one has presented very general (even fuzzy) concepts; while two have demonstrated what, in our opinion, seems to be an adequate learning level of those concepts that the didactical proposal approaches. Classification criteria for the level of generality of the concepts included both, conceptual accuracy and the number of criterial attributes associated with those concepts. Therefore, the fewer the number of criterial attributes associated to the concepts, the higher the level of concept generality.

Throughout the intervention, we could verify that because of the slight difference between the concepts of measurement (application of an experimental setup to obtain a value of dynamical variable) and determination (obtaining a single value for a set of identical objects prepared in the same state), students treated the concepts as if they were identical, under a high availability threshold[4]. Nevertheless, in the interview, when asked about concept differentiation (recognition of differences), some students pointed out to distinctions among them.

Five out of six students seem to have assimilated the concept of *state* as a differentiation of the concept of *dynamical variables*. The sequential organization of the content was constructed with the purpose of teaching the concept of *state* at the same level of inclusiveness as the concept of *dynamical variables*, but the students, who might have been more used to describing (in classical physics) the behavior of specific dynamical variables, have assimilated this concept more easily than the concept of *state*, to which it is subsumed. The only student who did not establish a direct relationship between these concepts used, in QM, examples of determination of state, including simultaneous knowledge of total energy and momentum, a determination that is hardly possible[5].

Four students seemed to have been able to understand the probabilistic aspect of quantum theory as something that is inherent to it. Nevertheless, one of them has presented hints of the occurrence of attribution of probability to an impossibility of determination of the state, which is considered as incorrect within the context of QM. This finding is similar to the one of Bao and Redish [16], and has hindered the assimilation of the concept of predictability (possibility of anticipating the state of the system at a different moment from the one in which it is known), though not necessarily interfering with the assimilation of a correlated concept, such as, for example, the causality concept (reinterpreted under the student's representation, that is, if the state cannot be known at that moment, it cannot be known in any other moment)[6]. One

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[4] Variable parameter presented by Ausubel to explain obliteration. It is a function of other variables such as: fatigue, stress, etc. As for the type of problem, the availability threshold depends on whether the question is of a recall or recognition type.

[5] The Hamiltonian operator, representing total energy, hardly ever commutes with momentum.

[6] The concepts of predictability and causality were dealt with during instruction as essential attributes to the concept of time evolution. A theory is predictive when, based on it, it is possible to know the state of the system we want to study. A theory is causal when it follows the principle

student has not evidenced his/her understanding of the concept of probability in QM in this sense, but we thought natural for him/her the treatment he/she has given to it.

We consider relevant here the facilitation of the learning process of the concepts of *Hamiltonian function* and *Hamiltonian operator* through the establishment of conceptual relationships between or among them and the concepts of interaction and energy, already known by the students. This relationship facilitated the understanding of some aspects of the Schrödinger equation, both the time-dependent and the time-independent, such as, for example, the one of the energy eigenvalues spectrum for some students (especially students 3, 4, and 5). It might be worth to emphasize the newness of the idea of "*formulation*" for the students.

The concept of superposition of states was the most heterogeneous one concerning the attributes the students assimilated, some of which the students had even erroneously/inadequately associated.

Two students strongly associated this concept with the indetermination between two dynamical variables, that is, in the determination of a variable associated to a quantum object as, for example, the position, the state of system is placed in a superposition of momentum eigenstates—a dynamic variable that is incompatible with the position. Student number six, on the other hand, associated that concept with the measurement concept through the relationship between measurement and state reduction. Student number two, shortly before instruction, conceived the concept of superposition of states as the existence of multiple dynamic variables (superposition of dynamic variables), improving the clearness of this concept along instruction. Student number four had a blurred understanding of this concept.

The concept of predictability in QM became more difficult for one of the students (student number two) who could not assimilate it. This difficulty might have derived from his/her conception of unpredictability of state (erroneous understanding of the probability concept). Student number six, because of his/her prior knowledge (mental representation of the reconstruction of the wave package), considered (implicitly) as being impossible to know the state of the system soon after it had been measured. For him/her, after a measurement, the reconstitution of the wave package occurs and, in temporal instances afterwards, the system is placed once again in a superposition of states. The origin of this prior knowledge comprises a topic that deserves further investigation.

Near the end of the instruction, we presented a Stern-Gerlach experiment type situation that comprised four items, which we reproduce below.

A silver atom with an initial value of x-component of the spin equal to $|\, s_x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}|\, s_z = \frac{\hbar}{2}\rangle + \frac{1}{\sqrt{2}}|\, s_z = -\frac{\hbar}{2}\rangle$*, which means, initially prepared in an eigenstate ... interacts with a non-uniform magnetic field oriented in the z direction. The spin state of the electron in an instant of time t, after the initial one is:* $|\psi(t)\rangle = \frac{e^{-\frac{i\omega t}{2}}}{\sqrt{2}}|\, s_z = \frac{\hbar}{2}\rangle + \frac{e^{+\frac{i\omega t}{2}}}{\sqrt{2}}|\, s_z = -\frac{\hbar}{2}\rangle$

a) If the dynamical variable s_x *is measured in the initial instant of time t=o, what will be found?*

b) If the dynamic variable s_z is measured in the initial instant of time $t = 0$, what will be found?

c) Knowing that the eigenstate is given by $|\, s_y = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}|\, s_z = \frac{\hbar}{2}\rangle + \frac{i}{\sqrt{2}}|\, s_z = -\frac{\hbar}{2}\rangle$*, when will the system have the value of* $s_y = \frac{\hbar}{2}$*?*

d) Will any dynamic variable have a defined value in the instant of time $t = \frac{2\pi}{\omega}$*? Justify your answer.*

Five students were able to carry out the task, however two of them did not manage to solve correctly the Stern-Gerlach experiment situation in the various instances. One of them believed that we could only state something about one of the variables, whilst nothing could be said about the other one (lack of understanding of the concepts of determination/ indetermination). Another student presented a similar condition in terms of confusion, because he/she did not display enough clarity in the concept of *determined* variable. The three remaining students (one of the six students only handed in only five out of the six requested tasks) were able to find the answers for items *a*, *b* and *d*, while item *c* was not adequately answered by any of the students. The concepts involved in that stuation need further investigation.

As for the assimilation of the ideas explicitly associated with the Schrödinger equation, it was possible to verify that four of the six students assimilated this equation in a way that could be related quite closely to the meaningful learning. Student number 2 assimilated the Schrödinger equation more mechanically than meaningfully, considering the arbitrary arguments he/she presented. A favorable aspect attained here was the more general understanding that this was the fundamental law of time evolution in QM.

Heisenberg's equation, which was approached in a much lesser instance in comparison to Schrödinger's equation, presented only the general attribute of being an alternative mechanism for the establishment of time evolution in QM. Some students, as student number six, could more clearly distinguish between these two formulations.

B. Operative Knowledge Analysis

For synthesis purposes, we will next comment on the possible operational invariants that are more common among students after presenting the relationships that contain the possible operational invariants mapped. We present possible theorems-in-action in the form of propositions that we have built and considered as equivalent in meaning to the ones students would construct.

of causality, which means that for each and every cause there is a subsequent effect. Time evolution in QM is constituted of those two aspects, but holding on to the reduction restriction of state that happens in measurement processes.

These propositions are constituted of concepts, written in italics, which are the inferred concepts-in-action.

Student number 1

- Theorem 1.1: In sequential measurements of the variables S_x and S_z, with Stern-Gerlach apparatus, the beam is always divided into two beams of equal intensity.
- Theorem 1.2: The *quantum state* carries information about the *intensity* of the beam. This *intensity* is associated with the probability of obtaining certain values of a dynamic variable.
- Theorem 1.3: If $|\psi\rangle = |a\rangle$, with $\hat{A}|a\rangle = a|a\rangle$ in the *measurement* of the *dynamic variable A*, alteration of the *state of the system* does not occur, nevertheless, if the system is in a *superposition* of states of the type $|\psi\rangle =$ *ca1a1+ca2 |a1*, there are *probabilities* associated to the obtention of the values a_1 and a_2.
- Theorem 1.4: Configurations of *quantum systems* are known through *probabilities*.

Student number 2

- Theorem 2.1: The *superposition of states* consists of the coexistence of values of *dynamic variables* about which we can get knowledge.
- Theorem 2.2: *Incompatible dynamic variables* are not susceptible to *simultaneous determination*. It is only possible, therefore, to obtain information about one of them.

Student number 3

- Theorem 3.1: In sequential measurement of the variables s_x and s_z, in Stern-Gerlach devices, the beam is always split in two beams of equal intensity.
- Theorem 3.2: The *quantum state* carries information about the *intensity* of the beam. Intensity is associated with the probability of obtaining a value of spin projection.
- Theorem 3.3: *Probability* is associated with the indeterminacy that is expressed by the *superposition of states*.
- Theorem 3.4: In quantum mechanics, *measurements* can modify the state of the system.
- Theorem 3.5: It is possible to know the *state of the system* along *time*, in spite of the *probabilities*. This behavior is dictated by Schrödinger's equation.

Student number 4

- Theorem 4.1: There is a *probability* associated to the *measurement* of *dynamic variables*.
- Theorem 4.2: If the *state of system* is expressed as a *superposition of eigenstates* of a *dynamic variable*, the *dynamic variable* in question has a defined value, because it is possible to know something about it.
- Theorem 4.3: Physics is *predictive*. In quantum mechanics we can *anticipate probabilities*.

Student number 5

- Theorem 5.1: In quantum mechanics, *measurements* can modify the state of the system.
- Theorem 5.2: In the Stern-Gerlach experiment, the *field direction* determines the direction of the beam division into two when measuring the … and … variables, being this split a product of the *destruction of prior information* that has been stored in a previous measurement.

Student number 6

- Theorem 6.1: If $|\psi\rangle = |a\rangle$, with $\hat{A}|a\rangle = a|a\rangle$ in the *measurement* of the *dynamic variable A*, alteration of the *state of the system* does not occur, nevertheless, if the system is in a *superposition* of states of the type $|\psi\rangle =$ *ca1a1+ca2 |a1*, there are *probabilities* associated to the obtention of the values a_1 and a_2.
- Theorem 6.2: *Time evolution* of the *state of system* places it in a *superposition of states*.
- Theorem 6.3: *Superposition of states* expresses *indetermination*.

We think that the discussion of the patterns in theorems-in-action and concepts-in-action shown in conceptualization processes, in the limitation verified in some of them, and in the mistakes present in others can be relevant here.

The theorem-in-action more clearly evidenced by students 1 and 3, who claimed "in sequential measurement of the variables s_x and s_z, in Stern-Gerlach devices, the beam is always split into two beams of equal intensity", is incomplete. This proposition will be valid in the case of sequential measurements of the components x and y, x and z, y and z of the spin, with an initial beam that had been prepared in an eigenstate $|s_x\rangle$, $|s_y\rangle$ or $|s_z\rangle$. If the field of the Stern-Gerlach apparatus were directed along a line that formed an angle (30 degrees for example) with the positive semi axis of x, in the (x,y) plan ($\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} \rightarrow tg\phi = \frac{B_y}{B_x}$), for instance, and the incident beams prepared, let's say, in an eigenstate $|s_z\rangle$, the resulting intensities of the beam after passing through the apparatus would not be equal. There was also evidences of the construction of this theorem by other students, but those appeared to be not so strong.

Another theorem, which students numbers 3 and 6 presented, claims that "*superposition of states* expresses *indetermination*", and student number 3, through probabilities, has more explicitly expressed this indetermination. This theorem, due to its very broad scope, does not specify whether, or not, this indetermination is associated to the variable that is to be measured. It is possible to have a hydrogen atom with a defined value of energy, that is, an atom in an eigenstate of energy and, nevertheless, it will be found in a superposition of eigenstates of linear momentum. The "indetermination" will be revealed when the momentum is measured, though not when we measure energy, which has been previously determined. Stern-Gerlach-experiment situations are presented to highlight this aspect, but it seems to be in a more specific scope than the theorems presented.

One theorem-in-action inferred qualitatively by two students (number 3 and 5) and quantitatively by two others (1 and 6) seems to demonstrate a cognitive construct similar to the *projection postulate*, which has as one of its consequences the change of the state of the quantum system with the act of measuring it. Students 3 and 5 presented this theorem as a possibility (measurement *can* change the state of system), while students 1 and 6, though having implicitly used the same theorem, presented it more directly. The theorem was enunciated as: "If $|\psi\rangle = |a\rangle$, with $\hat{A}|a\rangle = a|a\rangle$ in the *measurement* of the *dynamic variable A*, alteration of the *state of the system* does not occur, nevertheless, if the system is in a *superposition* of states of the type $|\psi\rangle = c_{a1}|a_1\rangle + c_{a2}\ |a_1\rangle$, there are *probabilities* associated to the obtention of the values a_1 and a_2". It should be emphasized that in the study of the measurement problem in Von Neumann's interpretation (or Princeton interpretation), the *projection postulate* constitutes a fundamental element that is adopted by most authors of QM texts.

An erroneous and, perhaps, very stable theorem attributes to probabilities a unique role in the knowledge of the configuration of the system (state). Students 1 and 4 implicitly presented this bias in the conceptualization process. Nonetheless, we know that the determination of the state requires knowledge of the probability *amplitudes.* The concept of probability amplitude, being more abstract (less concrete[7] and not less general) than the concept of probability has in this aspect a possible hint that the initial conceptualization by the students hardly takes it into consideration.

V. DISCUSSION

In the evidence of an argumentative discourse and conceptual relationships, both explicit and implicit, revealed in the analysis of the outcomes of instruction, it seems possible to suppose that most students (at least four of six) have developed a type of learning that is quite close to the meaningful end of the rote-meaningful learning continuum. Meaningful learning, nonetheless, does not necessarily qualify as a correct or complete type of learning, but it lasts longer than mechanical learning. As we could observe, both in the process of assimilation of knowledge in its predicative form and in the construction and use of possible operational invariants, incompleteness and misunderstandings might occur. During the investigation process it has been possible to verify, however, some encouraging and enlightening aspects.

- The crucial role of situations in the conceptualization process: They should be presented with the purpose of facilitating the use of the more adequate "template" of the theorems-in-action and the favorable use of concepts-in-action, so as to promote an early mastering of the conceptual field and, consequently, to attribute meaning to the concepts emerging in the proposal. Some of the inferred theorems-in-action can be reformulated by the students, when we emphasize certain characteristic subtleties that are present in the conceptualization process, such as the distinction between the concepts of probability and probability amplitude.
- The role of the negotiation of meanings, a potential facilitator of the meaningful learning of concepts that are compatible with the scientific knowledge. Classroom dialogue, which has demonstrated incompleteness and "flaws" in the conceptualization, can facilitate both progressive differentiation and integrative reconciliation of the content in the predicative form of knowledge, in the students' cognitive structure.
- The importance of recursiveness in the teaching-learning process of a conceptual field that can be so counter-intuitive as QM. The concepts of *state of a physical system*, *eigenstates,* and *superposition of states* always need to have their meanings reaffirmed in many contexts, because besides being concepts that are more distant from the students' prior knowledge (classical), they constitute an important part of the QM vocabulary.
- The importance of concepts such as *predictability* and *causality* that, when taught in association with those concepts aforementioned, can avoid the construction of epistemological obstacles when followed the listed indications (presentation of problem-situations and negotiation of meanings).
- Modification of the attributes of concepts such as *time evolution* that are already present, though in a fuzzy way, in the students' cognitive structure, which now incorporates the time-dependent Schrödinger's equation as the fundamental equation in the Non-Relativistic Quantum Mechanics.
- The importance assigned by the students to the concept of probability in detriment to the concept of probability amplitude. It must be stressed that this attribution can be a hindrance for a complete learning in Quantum Mechanics.

[7] Unlike the meaning used by Ausubel, we are dealing with the "physical" concreteness of the concept. The concept of probability amplitude has a physical interpretation less concrete than the concept of probability, which is directly associated to the intensity of a beam, to the number of counts of a detector, etc.

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