Jerk, curvature and torsion in motion of charged particle under electric and magnetic fields – Part II

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Abstract
The existence of the jerk vector and the resulting curvature and torsion were investigated in the motion of a charged particle under the influence uniform of electric and magnetic fields separately as well as under parallel electric and magnetic fields. In this second part, we investigate the motion of the charged particle under mutually perpendicular electric and magnetic fields and complete the investigation for the general case of electric and magnetic fields of arbitrary orientation. It is shown that under crossed electric and magnetic fields, the motion of the charged particle lies in a plane if the initial velocity has no component along the magnetic field. Also, the charged particle travels in a straight line if the initial velocity has no component along the electric field but has a component equal to the electromagnetic drift velocity in the direction orthogonal to both the electric and magnetic fields. In the general case of electric and magnetic fields of arbitrary orientation, the motion of the charged particle will lie in a plane if the electric field has no component along the magnetic field and if the initial velocity of the particle has no component in the direction of the magnetic field. The charged particle will travel in a straight line if the following conditions are met: (1) the electric field has no component parallel with the magnetic field; (2 & 3) the initial velocity has no components in the directions of the electric and magnetic fields; and (4) the initial velocity in the direction orthogonal to the electric and magnetic fields is equal to the electromagnetic drift velocity.

Keywords: Jerk vector, Curvature, Torsion.

Resumen
La existencia de un vector jerk y la curvatura y torsión resultante se investigaron en el movimiento de una partícula cargada bajo la influencia uniforme de los campos eléctricos y magnéticos separadamente así como bajo campos magnéticos y eléctricos paralelos. En esta segunda parte, investigamos el movimiento de la partícula cargada bajo campos magnéticos y eléctricos mutuamente perpendiculares y completar la investigación para el caso general de los campos eléctricos y magnéticos de orientación arbitraria. Esto muestra que bajo cruzados campos eléctricos y magnéticos, el movimiento de la partícula cargada se encuentra en un plano si la velocidad inicial no tiene ningún componente a lo largo del campo magnético. Además, la partícula cargada viaja en línea recta si la velocidad inicial no tiene ningún componente a lo largo del campo eléctrico, pero tiene un elemento igual a la velocidad de arrastre electromagnética en la dirección ortogonal a ambos campos eléctricos y magnéticos. En el caso general de los campos eléctricos y magnéticos de orientación arbitraria, el movimiento de la partícula cargada se encontrará en un plano si el campo eléctrico no tiene ningún componente a lo largo del campo magnético y si la velocidad inicial de la partícula no tiene componente en la dirección del campo magnético. La partícula cargada se desplaza en línea recta si se cumplen las siguientes condiciones: (1) el campo eléctrico no tiene componente paralelo con el campo magnético; (2 y3) la velocidad inicial no tiene componentes en las direcciones de los campos eléctricos y magnéticos; y (4) la velocidad inicial en la dirección ortogonal a los campos eléctricos y magnéticos es igual a la velocidad de arrastre electromagnética.

Palabras clave: Vector Jerk, Curvatura, Torsión.

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I. INTRODUCTION

One of the relatively neglected topics in physics is the *jerk vector*. Formerly known as the *second acceleration*, it is the derivative of the acceleration vector, or the third derivative of the position vector, with respect to time. Nonetheless, the jerk vector has been studied in simple harmonic motion [1], uniform circular motion [2], Keplerian motion [1, 3], projectile motion [4], and the motion of a charged particle under uniform parallel electric and magnetic fields [5]. In this paper, we investigate the jerk vector on a charged particle under mutually perpendicular electric and magnetic fields, as well as the general problem of charged particle motion under electric and magnetic fields of arbitrary orientation.

Two other concepts related to a curve and seldom mentioned in physics are the *curvature* and *torsion*. The curvature is the arc-rate of turning of the tangent vector in a plane, whereas the torsion, formerly called the *second curvature*, is the arc-rate of turning of the tangent out of the plane [6, 7]. These concepts of differential geometry are well-suited to dynamical problems when the curve under consideration is the trajectory of a particle. If the first three derivatives of the position vector in time, viz., the velocity, acceleration and the jerk vectors are $\vec{v}$, $\vec{a}$ and $\vec{j}$, respectively, then the curvature κ and torsion τ are given by [5]:

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}, \qquad (1)$$

and

$$\tau = \frac{|\vec{v} \circ (\vec{a} \times \vec{j})|}{|\vec{v} \times \vec{a}|^2}. \qquad (2)$$

From the definitions of curvature and torsion, Eqs. (1) and (2) suggest that a sufficient condition for the particle trajectory to lie in a plane is $\vec{v} \circ (\vec{a} \times \vec{j}) = 0$, whereas a sufficient condition for rectilinear motion of the particle is given by $\vec{v} \times \vec{a} = \vec{0}$.

II. MOTION OF CHARGED PARTICLE UNDER CROSSED UNIFORM ELECTRIC AND MAGNETIC FIELDS

The equation of motion of a charged particle of mass m and electric charge q under the electric field $\vec{E}$ and magnetic field $\vec{B}$ is given by the Lorentz equation. In Gaussian system of units, we have [8]:

$$m\frac{d\vec{v}}{dt} = q\vec{E} + q\vec{v} \times \vec{B}. \qquad (3)$$

In our first example, we consider the motion of a charged particle q (>0) under the combined actions of mutually perpendicular electric and magnetic fields. Let $\vec{E} = E\hat{y}$ and $\vec{B} = B\hat{x}$ (Fig. 1).

Then the equations of motion are the following:

$$\frac{dv_x}{dt} = \Omega v_y, \qquad (4)$$

$$\frac{dv_y}{dt} = \Lambda - \Omega v_x, \qquad (5)$$

and

$$\frac{dv_z}{dt} = 0, \qquad (6)$$

where $\Omega = qB/m$ is the gyro-frequency and $\Lambda = qE/m$.

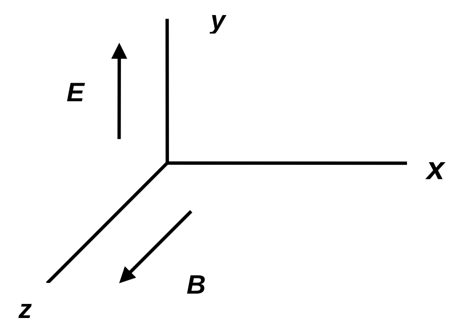

FIGURE 1. Charged particle in crossed electric and magnetic fields.

Eqs. (5) and (6) are coupled differential equations. By differentiation and elimination, one gets:

$$\frac{d^2v_x}{dt^2} + \Omega^2\left(v_x - \frac{\Lambda}{\Omega}\right) = 0, \qquad (7)$$

and

$$\frac{d^2v_y}{dt^2} + \Omega^2 v_y = 0. \qquad (8)$$

Eqs. (7) and (8) have the general solutions:

$$v_x = \frac{\Lambda}{\Omega} + A\cos\Omega t + B\sin\Omega t, \qquad (9)$$

and

$$v_y = C\cos\Omega t + D\sin\Omega t, \qquad (10)$$

whereas Eq. (6) integrates directly to

$$v_z = const. \qquad (11)$$

If the initial components of the velocity in the *x*-, *y*- and *z*-directions be v_{x0}, v_{y0} and v_{z0}, respectively, then

$$v_x = \frac{\Lambda}{\Omega} + \left(v_{x0} - \frac{\Lambda}{\Omega}\right)\cos\Omega t + v_{y0}\sin\Omega t, \qquad (12)$$

$$v_y = -\left(v_{x0} - \frac{\Lambda}{\Omega}\right)\sin\Omega t + v_{y0}\cos\Omega t, \qquad (13)$$

and

$$v_z = v_{z0}\,. \tag{14}$$

By successive differentiations, we obtain the acceleration and the jerk vector components:

$$a_x = -\left(v_{x0} - \frac{\Lambda}{\Omega}\right)\Omega\sin\Omega t + v_{y0}\Omega\cos\Omega t\,, \tag{15}$$

$$a_y = -\left(v_{x0} - \frac{\Lambda}{\Omega}\right)\Omega\cos\Omega t - v_{y0}\Omega\sin\Omega t\,, \tag{16}$$

$$a_z = 0\,, \tag{17}$$

$$j_x = -\left(v_{x0} - \frac{\Lambda}{\Omega}\right)\Omega^2\cos\Omega t - v_{y0}\Omega^2\sin\Omega t\,, \tag{18}$$

$$j_y = \left(v_{x0} - \frac{\Lambda}{\Omega}\right)\Omega^2\sin\Omega t - v_{y0}\Omega^2\cos\Omega t\,, \tag{19}$$

and

$$j_z = 0\,. \tag{20}$$

From Eqs. (12–20), we get:

$$\vec{a}\times\vec{j} = -\Omega^3\left[\left(v_{x0} - \frac{\Lambda}{\Omega}\right)^2 + {v_{y0}}^2\right]\hat{z}\,, \tag{21}$$

$$\vec{v}\circ\left(\vec{a}\times\vec{j}\right) = -\Omega^3\left[\left(v_{x0} - \frac{\Lambda}{\Omega}\right)^2 + {v_{y0}}^2\right]v_{z0}\,, \tag{22}$$

and

$$\vec{v}\times\vec{a} = v_{z0}\Omega\left[v_{y0}\sin\Omega t + \left(v_{x0} - \frac{\Lambda}{\Omega}\right)\cos\Omega t\right]\hat{x}$$

$$+ v_{z0}\Omega\left[v_{y0}\cos\Omega t - \left(v_{x0} - \frac{\Lambda}{\Omega}\right)\sin\Omega t\right]\hat{y}$$

$$- \Lambda\left[v_{y0}\sin\Omega t + \left(v_{x0} - \frac{\Lambda}{\Omega}\right)\cos\Omega t\right]\hat{z}$$

$$+ \Omega\left[{v_{y0}}^2 + \left(v_{x0} - \frac{\Lambda}{\Omega}\right)^2\right]\hat{z}\,. \tag{23}$$

Eq. (22) in conjunction with Eq. (2) indicates that a sufficient condition for the torsion to be zero is $v_{z0} = 0$. Recalling that torsion is the rate of turning of the tangent vector out of the plane of motion, this indicates that *if the initial velocity has no component along the magnetic field* (*z-direction*), *then the trajectory of the charged particle lies in the x-y plane*.

Eqs. (23) in conjunction with Eq. (1) indicates that the sufficient conditions for the curvature to be zero are given by $v_{y0} = 0$ and $v_{x0} = \Lambda/\Omega = E/B$. Recalling that the curvature is the rate of turning of the tangent vector in the plane of motion and recognizing that E/B represents the magnitude of the electromagnetic drift velocity, this indicates that *the charged particle will travel in a straight line if the initial velocity has no component along the electric field* (*y-direction*) *but has a component equal to the drift velocity in the direction orthogonal to the electric and magnetic fields* (*x-direction*).

III. CHARGE PARTICLE TRAJECTORIES

It is instructive to study the trajectory of the charged particle under crossed uniform electric and magnetic fields, particularly for interesting special cases.

A. $v_{z0} = 0$

We have seen that when the initial velocity has no component along the magnetic field (*z*-direction), the trajectory of the charged particle lies in the *x*-*y* plane and the resulting torsion is zero.

B. $v_{z0} = 0$ and $v_{y0} = 0$

If the initial velocity of the charged particle possesses no components along the electric field (*y*-direction) and the magnetic field (*z*-direction), then Eqs. (12) and (13) assume the simpler forms

$$v_x = \frac{\Lambda}{\Omega} + \left(v_{x0} - \frac{\Lambda}{\Omega}\right)\cos\Omega t\,, \tag{24}$$

And

$$v_y = -\left(v_{x0} - \frac{\Lambda}{\Omega}\right)\sin\Omega t\,. \tag{25}$$

If further, the particle is initially located at the origin (*i.e.*, at $t = 0$, $x = 0$ and $y = 0$), then Eqs. (24) and (25) integrate to

$$x = \frac{\Lambda}{\Omega}t + \frac{1}{\Omega}\left(v_{x0} - \frac{\Lambda}{\Omega}\right)\sin\Omega t\,, \tag{26}$$

and

$$y = -\frac{v_{x0}}{\Omega} + \frac{\Lambda}{\Omega^2} - \frac{\Lambda}{\Omega^2}\left(1 - v_{x0}\frac{\Omega}{\Lambda}\right)\cos\Omega t\,. \tag{27}$$

Eqs. (26) and (27) are of the form

$$x = a\theta - b\sin\theta, \quad (28)$$

$$y + c = a - b\cos\theta, \quad (29)$$

where

$$a = \frac{\Lambda}{\Omega^2}, \quad (30)$$

$$b = \frac{\Lambda}{\Omega^2}\left(1 - v_{x0}\frac{\Omega}{\Lambda}\right), \quad (31)$$

$$c = \frac{v_{x0}}{\Omega}, \quad (32)$$

and

$$\theta = \Omega t. \quad (33)$$

Eqs. (28) and (29) are the parametric equations of a *trochoid* in the *x-y* plane [9]. In this case, it is a curve generated by a point fixed at a distance *b* from the center of a circle of radius *a* on the straight line $y = 0$ [9].

C. $v_{z0} = 0$, $v_{y0} = 0$ and $v_{x0} < 0$

In this case $b > a$ and the trajectory is a *prolate cycloid* [9].

D. $v_{z0} = 0$, $v_{y0} = 0$ and $v_{x0} = 0$

In this case, Eqs. (28) and (29) reduce to those of a regular *cycloid*:

$$x = a(\theta - \sin\theta), \quad (34)$$

and

$$y + c = a(1 - \cos\theta). \quad (35)$$

E. $v_{z0} = 0$, $v_{y0} = 0$ and $v_{x0} > 0$

In this case $b < a$ and the trajectory is a *curtate cycloid* [9].

F. $v_{z0} = 0$, $v_{y0} = 0$ and $v_{x0} = \Lambda/\Omega = E/B$

This is the most interesting case mentioned earlier when the initial x-component of the velocity is equal to the electromagnetic drift velocity. Eqs. (24) and (25) reduce to

$$v_x = \frac{\Lambda}{\Omega}, \quad (36)$$

and

$$v_y = 0, \quad (37)$$

which integrate to

$$x = x_0 + \frac{\Lambda}{\Omega}t, \quad (38)$$

and

$$y = y_0. \quad (39)$$

The trajectory is a straight line parallel with the *x*-axis for which both the curvature and torsion are equal to zeros.

As illustrations of cases C, D, E and F above, we consider the following initial velocity components in addition to $v_{z0} = 0$ and $v_{y0} = 0$: (1) $v_{x0} = -.5\Lambda/\Omega$; (2) $v_{x0} = 0$; (3) $v_{x0} = .5\Lambda/\Omega$ and (4) $v_{x0} = \Lambda/\Omega$. Their trajectories are shown in Fig. 2 marked as 1 (prolate cycloid); 2 (cycloid); 3 (curtate cycloid) and 4 (straight line), respectively. The trajectories straighten out as v_{x0} increases.

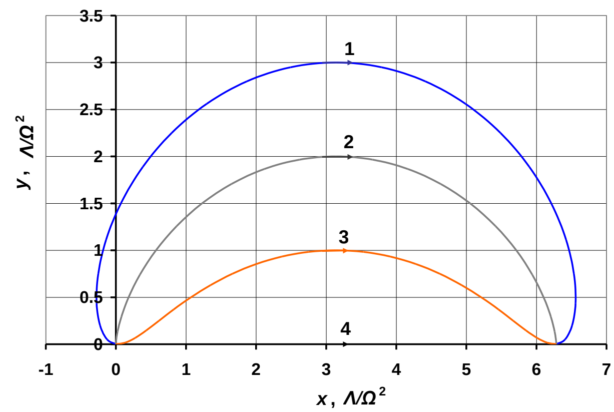

FIGURE 2. Trajectories of charged particle *q* (>0) for different initial velocity components.

IV. MOTION OF CHARGED PARTICLE UNDER UNIFORM ELECTRIC AND MAGNETIC FIELDS OF ARBITRARY ORIENTATION

For the sake of completeness, we next consider the motion of a charged particle *q* (>0) under the actions of uniform electric and magnetic fields of arbitrary orientation. Without the loss of generality, we can choose $\vec{B} = B\hat{z}$ and $\vec{E} = E_{\perp}\hat{y} + E_{II}\hat{z}$ [10] as shown in Fig. 3.

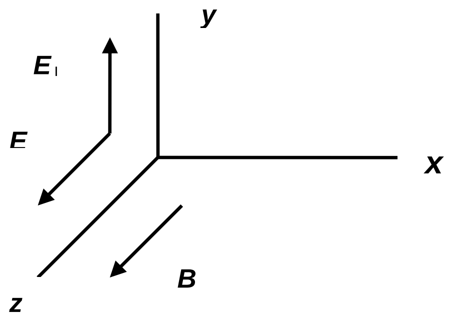

FIGURE 3. Charged particle in electric and magnetic fields of general orientation.

Then the equations of motion are the following:

$$\frac{dv_x}{dt} = \Omega v_y \,, \tag{40}$$

$$\frac{dv_y}{dt} = \Lambda_{\perp} - \Omega v_x \,, \tag{41}$$

and

$$\frac{dv_z}{dt} = \Lambda_{\parallel} \,, \tag{42}$$

where $\Omega = qB/m$, $\Lambda_{\perp} = qE_{\perp}/m$ and $\Lambda_{\parallel} = qE_{\parallel}/m$.

Eqs. (40) and (41) are coupled differential equations. By differentiation and elimination, one gets

$$\frac{d^2v_x}{dt^2} + \Omega^2\left(v_x - \frac{\Lambda_{\perp}}{\Omega}\right) = 0 \,, \tag{43}$$

and

$$\frac{d^2v_y}{dt^2} + \Omega^2 v_y = 0 \,. \tag{44}$$

Eqs. (43) and (44) have the general solution

$$v_x = \frac{\Lambda_{\perp}}{\Omega} + A\cos\Omega t + B\sin\Omega t \,, \tag{45}$$

and

$$v_y = C\cos\Omega t + D\sin\Omega t \,, \tag{46}$$

where as Eq. (42) integrates directly to

$$v_z = \Lambda_{\parallel} t + const. \tag{47}$$

If the initial components of the velocity in the *x*-, *y*- and *z*-directions be v_{x0}, v_{y0} and v_{z0}, respectively, then

$$v_x = \frac{\Lambda_{\perp}}{\Omega} + \left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\cos\Omega t + v_{y0}\sin\Omega t \,, \tag{48}$$

$$v_y = -\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\sin\Omega t + v_{y0}\cos\Omega t \,, \tag{49}$$

and

$$v_z = v_{z0} + \Lambda_{\parallel} t \,. \tag{50}$$

By successive differentiations, we obtain the acceleration and the jerk vector components:

$$a_x = -\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\Omega\sin\Omega t + v_{y0}\Omega\cos\Omega t \,, \tag{51}$$

$$a_y = -\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\Omega\cos\Omega t - v_{y0}\Omega\sin\Omega t \,, \tag{52}$$

$$a_z = \Lambda_{\parallel} \,, \tag{53}$$

$$j_x = -\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\Omega^2\cos\Omega t - v_{y0}\Omega^2\sin\Omega t \,, \tag{54}$$

$$j_y = \left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\Omega^2\sin\Omega t - v_{y0}\Omega^2\cos\Omega t \,, \tag{55}$$

and

$$j_z = 0 \,. \tag{56}$$

From Eqs. (51–56), we get:

$$\begin{aligned}\vec{a}\times\vec{j} = &-\Lambda_{\parallel}\Omega^2\left[\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\sin\Omega t - v_{y0}\cos\Omega t\right]\hat{x} \\ &+ \Lambda_{\parallel}\Omega^2\left[\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\cos\Omega t + v_{y0}\sin\Omega t\right]\hat{y} \\ &- \Omega^3\left[\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)^2 + v_{y0}^{\;2}\right]\hat{z} \,,\end{aligned} \tag{57}$$

and

$$\begin{aligned}\vec{v}\circ\left(\vec{a}\times\vec{j}\right) = &-\Lambda_{\parallel}\Omega\Lambda_{\perp}\left[\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)\sin\Omega t - v_{y0}\cos\Omega t\right] \\ &- 2\Lambda_{\parallel}\Omega^2\left[\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)^2 + v_{y0}^{\;2}\right]\sin\Omega t\cos\Omega t \\ &- 2\Lambda_{\parallel}\Omega^2\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)v_{y0}\left(\sin^2\Omega t - \cos^2\Omega t\right) \\ &- \Omega^3\left[\left(v_{x0} - \frac{\Lambda_{\perp}}{\Omega}\right)^2 + v_{y0}^{\;2}\right]\left(\Lambda_{\parallel} t + v_{z0}\right) .\end{aligned} \tag{58}$$

Eqs. (2) and (58) indicate that $\Lambda_{II} = 0$ (*i.e.*, $E_{II} = 0$) and $v_{z0} = 0$ comprise sufficient conditions for the torsion to be zero. In other words, *the motion of the charged particle will be in a plane if the electric field has no component parallel with the magnetic field and if the initial velocity of the particle has no component in the direction of the magnetic field*.

Further, we have

$$(\vec{v}\times\vec{a})_x = \Lambda_{II}\left(v_{x0}-\frac{\Lambda_\perp}{\Omega}\right)(\Omega t\cos\Omega t-\sin\Omega t) + \Lambda_{II} v_{y0}(\cos\Omega t+\Omega t\sin\Omega t) + v_{z0}\left[v_{y0}\Omega\sin\Omega t+\left(v_{x0}-\frac{\Lambda_\perp}{\Omega}\right)\Omega\cos\Omega t\right], \quad (59)$$

$$(\vec{v}\times\vec{a})_y = \Lambda_{II}\left[-\frac{\Lambda_\perp}{\Omega}+v_{y0}(\Omega t\cos\Omega t-\sin\Omega t)\right] - \Lambda_{II}\left(v_{x0}-\frac{\Lambda_\perp}{\Omega}\right)(\Omega t\sin\Omega t+\cos\Omega t) + v_{z0}\left[-\left(v_{x0}-\frac{\Lambda_\perp}{\Omega}\right)\Omega\sin\Omega t+v_{y0}\Omega\cos\Omega t\right], \quad (60)$$

and

$$(\vec{v}\times\vec{a})_z = -\Lambda_\perp\left[v_{y0}\sin\Omega t+\left(v_{x0}-\frac{\Lambda_\perp}{\Omega}\right)\cos\Omega t\right] - \Omega\left[\left(v_{x0}-\frac{\Lambda_\perp}{\Omega}\right)^2+v_{y0}^{\,2}\right]. \quad (61)$$

Eq. (1) in conjunction with Eqs. (59–61) indicates that sufficient conditions for the curvature to be zero are given by: $\Lambda_{II} = 0$ (*i.e.*, $E_{II} = 0$); $v_{z0} = 0$; $v_{y0} = 0$; and $v_{x0} = \Lambda_\perp/\Omega = E_\perp/B$. This means *that the charged particle will travel in a straight line if the following conditions are met:* (1) *The electric field has no component parallel with the magnetic field;* (2 & 3) *The initial velocity had no components in the directions of the electric and magnetic fields; and* (4) *The initial velocity in the direction orthogonal to the electric and magnetic fields is equal to the electromagnetic drift velocity*.

This concludes the general problem of the motion of a charged particle under constant electric and magnetic fields of arbitrary orientation. All other examples follow as special cases of the general problem. The first example in this paper is realized when $v_{II} = 0$, whereas the case of parallel electric and magnetic fields [5] is obtained when $v_\perp = 0$.

V. CONCLUSIONS

The topics of jerk, curvature and torsion are not part of the normal curriculum and are seldom discussed in the literature. This paper, along with its predecessor, demonstrates that they are useful concepts which can be applied to common dynamical problems and valuable information can be gleaned from them.

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