Finding the maximum magnitude response (gain) of second-order filters without calculus

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Abstract

The maximum gain (with respect to frequency) of second-order filters such as low-pass, high-pass, band-pass, low-pass notch and high-pass notch filters is derived without using calculus. Our method uses the fact that the square of the magnitude response (gain) for these filters can be written as a positive constant divided by a quadratic function of frequency. Hence, the gain is maximized when the denominator of the gain is minimized, which is easily achieved without calculus, as the denominator is parabolic.

Keywords: Filters, Maximum without Calculus, Maximum Gain.

Resúmen

Derivamos, sin usar cálculo, la ganancia máxima (con respecto a la frecuencia) de filtros de segundo orden, como filtros de paso bajo, paso alto, pasabanda, muesca de paso bajo, y muesca de paso alto. Nuestro método se basa en el hecho de que el cuadrado de la respuesta de magnitud (ganancia) para estos filtros se puede escribir como una constante positiva dividida por una función cuadrática de la frecuencia. Por consiguiente, la ganancia se maximiza cuando el denominador se minimiza, lo que se hace fácilmente sin cálculo porque el denominador es parabólico.

Palabras cave: Filtros, Máximo sin cálculo, Ganancia máxima.

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I. INTRODUCTION

Second-order filters are ubiquitous in electronics. One important feature of these filters is the maximum magnitude response (gain) attainable by the filter, whether they are low-pass, high-pass, band-pass, low-pass notch, or high-pass notch filters. Such filters are studied by many authors (please see *e.g.*, pp. 15 and 16 of [1] and [2]) where, conventionally, this maximum value is found with calculus, or indeed simply stated without derivation. However, this puts the curious student who has not yet had the chance to study calculus at a disadvantage. Presently, he or she has no choice but to accept the equations for the maximum gain without any understanding as to their origins. Fortunately, as we show in this paper, it is straightforward to derive this maximum gain without calculus.

II. GAIN OF SELECTED SECOND-ORDER FILTERS

In this section, the gain of selected second-order filters is given.

A. Low-Pass Filter

The transfer function of a second-order low-pass filter is given by

$$T(s)=\frac{a_0}{s^2+(\omega_p/Q)s+\omega_p^2}, \tag{1}$$

where a_0, ω_p and Q are parameters of the filter, and $s=i\omega$, with $i=\sqrt{-1}$, and ω is the angular frequency of the applied sine-wave.

The gain of the second-order low-pass filter is simply the magnitude of Eq. (1), *i.e.*

$$|T(\omega)|=\frac{|a_0|}{\sqrt{\left(\omega_p^2-\omega^2\right)^2+\left(\omega_p\omega/Q\right)^2}}. \tag{2}$$

A sketch of Eq. (2) is shown in Fig. 1, where the maximum gain, $|T|_{max}$, and the frequency at which it occurs, ω_{max}, are clearly identified.

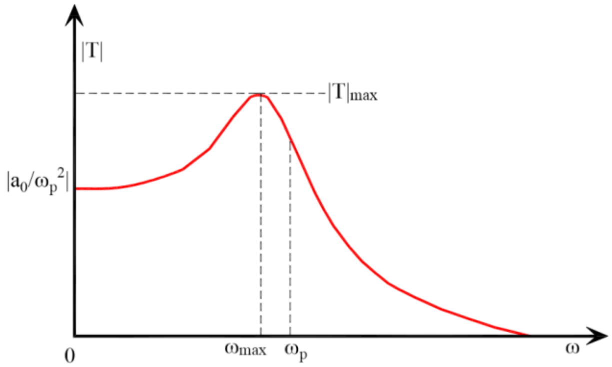

FIGURE 1. Magnitude response (gain) $|T(\omega)|$ of a second-order low-pass filter.

B. High-Pass Filter

The transfer function of a second-order high-pass filter is given by

$$T(s) = \frac{a_2 s^2}{s^2+(\omega_p/Q)s+\omega_p^2}. \quad (3)$$

The gain of the second-order high-pass filter is the magnitude of Eq. (3), *i.e.*

$$|T(\omega)| = \frac{|a_2|\omega^2}{\sqrt{\left(\omega_p^2-\omega^2\right)^2+\left(\omega_p\omega/Q\right)^2}}. \quad (4)$$

A sketch of Eq. (4) is shown in Fig. 2, where the maximum gain, $|T|_{max}$, and the frequency at which it occurs, ω_{max}, are clearly identified.

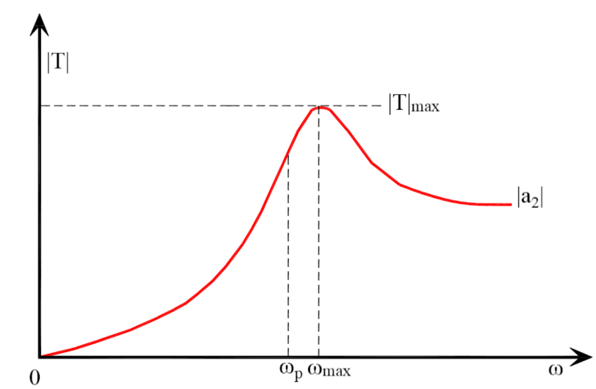

FIGURE 2. Magnitude response (gain) $|T(\omega)|$ of a second-order high-pass filter.

C. Band-Pass Filter

The transfer function of a second-order band-pass filter is given by

$$T(s) = \frac{a_1 s}{s^2+(\omega_p/Q)s+\omega_p^2}. \quad (5)$$

The gain of the second-order band-pass filter is the magnitude of Eq. (5), *i.e.*

$$|T(\omega)| = \frac{|a_1|\omega}{\sqrt{\left(\omega_p^2-\omega^2\right)^2+\left(\omega_p\omega/Q\right)^2}}. \quad (6)$$

A sketch of Eq. (6) is shown in Fig. 3, where the maximum gain, $|T|_{max}$, and the frequency at which it occurs, $\omega_{max} = \omega_p$, are clearly identified.

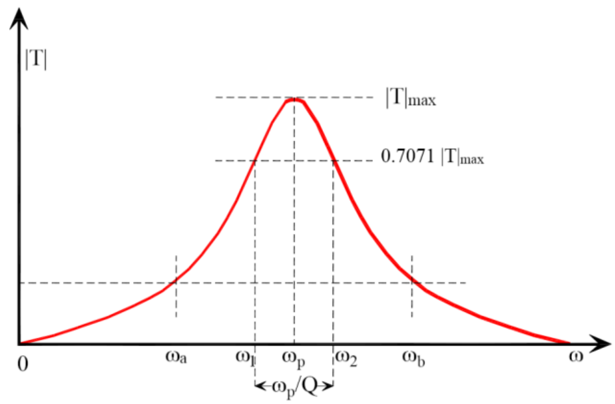

FIGURE 3. Magnitude response (gain) $|T(\omega)|$ of a second-order band-pass filter.

D. Low-Pass Notch and High-Pass Notch Filters

The transfer function of a second-order low-pass notch filter and the transfer function of a second order high-pass notch filter are identical and are given by

$$T(s) = \frac{a_2\left(s^2+\omega_z^2\right)}{s^2+(\omega_p/Q)s+\omega_p^2}. \quad (7)$$

Note that $\omega_z > \omega_p$ for the low-pass notch filter, whereas for the high-pass filter, $\omega_p > \omega_z$.

Using Eq. (7), the gain is easily found to be

$$|T(\omega)| = \frac{|a_2|\left|-\omega^2+\omega_z^2\right|}{\sqrt{\left(\omega_p^2-\omega^2\right)^2+\left(\omega_p\omega/Q\right)^2}}. \quad (8)$$

A sketch of Eq. (8) is shown in Fig. 4 for the low-pass notch filter and in Fig. 5 for the high-pass notch filter.

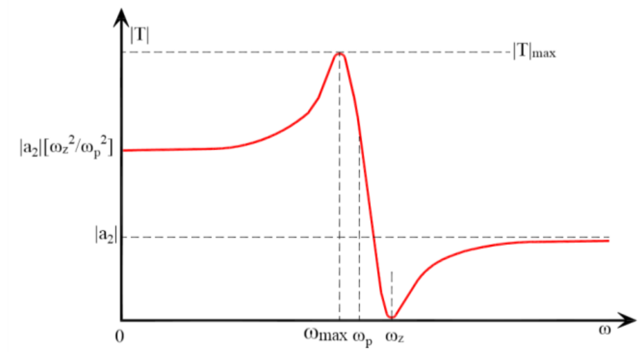

FIGURE 4. Magnitude response (gain) $|T(\omega)|$ of a second-order low-pass notch filter.

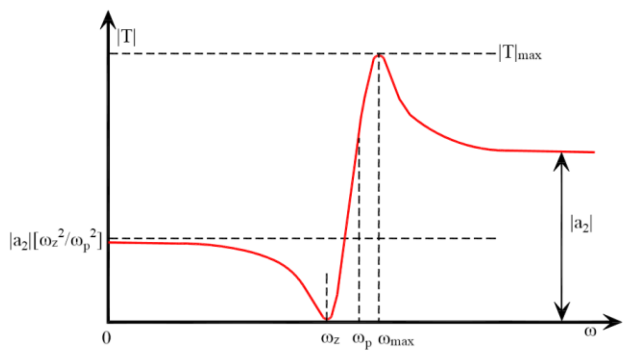

FIGURE 5. Magnitude response (gain) $|T(\omega)|$ of a second-order high-pass notch filter.

III. DETERMINING THE MAXIMUM GAIN OF SELECTED SECOND-ORDER FILTERS WITHOUT USING CALCULUS

Now that the gains of selected second-order filters have been stated, we can show how the maximum gain can be determined without using calculus. To accomplish this, we show in the appendices that, for all filters discussed here, the square of the gain can be written as a positive constant divided by a quadratic function of frequency, as given in Eq. (9):

$$|T(\omega)|^2 = \frac{A}{u^2+B}, \tag{9}$$

where A, B are positive constants and u is a function of angular frequency ω.

Hence, to maximize the gain with respect to frequency, we simply minimize the denominator, *i.e.*, we set $u = 0$. Therefore, the maximum gain is given by

$$|T(\omega)|_{max} = \sqrt{\frac{A}{B}}. \tag{10}$$

A. Low-Pass Filter

The square of the gain of the second-order low-pass filter, as shown in Appendix A, can be rewritten as Eq. (9) with $A = a_0^2$, $u = \omega^2 - \left(1 - \frac{1}{2Q^2}\right)\omega_p^2$ and $B = \omega_p^4 - \left(1 - \frac{1}{2Q^2}\right)^2 \omega_p^4 = \frac{1}{Q^2}\left(1 - \frac{1}{4Q^2}\right)\omega_p^4$.

As explained above, Eq. (9) is a maximum when its denominator is minimum, *i.e.*, when $u = 0 = \omega_{max}^2 - \left(1 - \frac{1}{2Q^2}\right)\omega_p^2$.

Hence,

$$\omega_{max} = \sqrt{1 - \frac{1}{2Q^2}}\,\omega_p. \tag{11}$$

Additionally, the maximum gain is given by Eq. (10), *i.e.*,

$$|T|_{max} = \frac{|a_0|}{\sqrt{B}} = \frac{|a_0|Q}{\omega_p^2\sqrt{1-\frac{1}{4Q^2}}}. \tag{12}$$

Fortunately, Eq. (11) and Eq. (12) are the same as those derived with calculus.

B. High-Pass Filter

As shown in Appendix B, the square of the gain of the second-order high-pass filter can be rewritten as Eq. (9) with $A = a_2^2$, $u = \left(\frac{\omega_p}{\omega}\right)^2 - \left(1 - \frac{1}{2Q^2}\right)$ and $B = 1 - \left(1 - \frac{1}{2Q^2}\right)^2$.

Recall that Eq. (9) is a maximum when its denominator is minimum, *i.e.*, when $u = \left(\frac{\omega_p}{\omega_{max}}\right)^2 - \left(1 - \frac{1}{2Q^2}\right) = 0$.

Hence,

$$\omega_{max} = \omega_p \Big/ \sqrt{1 - \frac{1}{2Q^2}}. \tag{13}$$

Additionally, the maximum gain is given by Eq. (10), *i.e.*, as is also shown in Appendix B,

$$|T|_{max} = \frac{|a_2|}{\sqrt{B}} = \frac{|a_2|Q}{\sqrt{1-\frac{1}{4Q^2}}}. \tag{14}$$

Fortunately, Eq. (13) and Eq. (14) are the same as those derived with calculus.

C. Band-Pass Filter

Similarly, as shown in Appendix C, the square of the gain of the second-order band-pass filter can be rewritten as Eq. (9) with $A = a_1^2$, $u = \frac{\omega_p^2 - \omega^2}{\omega}$ and $B = \frac{\omega_p^2}{Q^2}$.

Furthermore, Eq. (9) is a maximum when its denominator is minimum, *i.e.*, when $u=\frac{\omega_p{}^2}{\omega_{\max}}-\omega_{\max}=0.$

Hence,

$$\omega_{max}=\omega_p. \quad (15)$$

Additionally, the maximum gain is given by Eq. (10), *i.e.*,

$$|T|_{max}=\frac{|a_1|}{\sqrt{B}}=\frac{|a_1|Q}{\omega_p}. \quad (16)$$

Fortunately, Eq. (15) and Eq. (16) are the same as those derived with calculus.

D. Low-Pass Notch and High-Pass Notch Filters

The square of the gain of the second-order low-pass notch and high-pass notch filter, as shown in Appendix D, can be rewritten as Eq. (9) with $A=a_2^2$, $u=\frac{\sqrt{D}}{y}+\frac{C}{\sqrt{D}}$, where

$C=1-k-\frac{1}{2Q^2}$, $D=(1-k)^2+\frac{k}{Q^2}$, $y=k-\left(\frac{\omega}{\omega_p}\right)^2$,

$k=\left(\frac{\omega_z}{\omega_p}\right)^2$, and $B=1-\frac{C^2}{D}$.

Recall that Eq. (9) is a maximum when its denominator is minimum, *i.e.*, when $u=\frac{\sqrt{D}}{k-\left(\frac{\omega_{\max}}{\omega_p}\right)^2}+\frac{C}{\sqrt{D}}=0.$

Hence, as shown in Appendix D,

$$\omega_{\max}=\omega_p\sqrt{\frac{\left(\frac{\omega_z}{\omega_p}\right)^2\left[1-\frac{1}{2Q^2}\right]-1}{\left(\frac{\omega_z}{\omega_p}\right)^2+\frac{1}{2Q^2}-1}}. \quad (17)$$

Additionally, the maximum gain is given by Eq. (10), *i.e.*, as is also shown in Appendix D,

$$|T(\omega)|_{\max}=|a_2|Q\sqrt{\frac{\left(1-\left(\frac{\omega_z}{\omega_p}\right)^2\right)^2+\frac{1}{Q^2}\left(\frac{\omega_z}{\omega_p}\right)^2}{1-\frac{1}{4Q^2}}}. \quad (18)$$

Fortunately, Eq. (17) and Eq. (18) are the same as those derived with calculus.

IV. CONCLUSIONS

We have shown that the square of the magnitude response (gain) of the low-pass, high-pass, band-pass, low-pass notch and high-pass notch filters can be written as a positive constant divided by a quadratic function of frequency, as given in Eq. (9). Using this fact, we have shown how the maximum gain of these filters can be found without calculus, which should be of benefit to the pre-calculus student.

REFERENCES

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APPENDIX A

In this appendix, we show that the square of the gain of the second-order low-pass filter is given by Eq. (9), and the maximum gain is given by Eq. (12).

Recall that the gain of a second order low-pass filter is given by Eq. (2). Expanding the brackets of this equation produces

$$|T(\omega)|=\frac{|a_o|}{\sqrt{\omega_p{}^4-2\omega_p{}^2\omega^2+\omega^4+\frac{\omega^2\omega_p{}^2}{Q^2}}}=\frac{|a_o|}{\sqrt{\omega_p{}^4-\left(-\frac{1}{Q^2}+2\right)\omega^2\omega_p{}^2+\omega^4}}. \quad \text{(A1)}$$

Completing the square of the quadratic in Eq. (A1) gives

$$|T(\omega)|=\frac{|a_o|}{\sqrt{\left(\omega^2-\left(1-\frac{1}{2Q^2}\right)\omega_p{}^2\right)^2+\omega_p{}^4-\left(1-\frac{1}{2Q^2}\right)^2\omega_p{}^4}}. \quad \text{(A2)}$$

Let $u=\omega^2-\left(1-\frac{1}{2Q^2}\right)\omega_p{}^2$ and $B=\omega_p{}^4-\left(1-\frac{1}{2Q^2}\right)^2\omega_p{}^4$, so that Eq. (A2) becomes:

$$|T(\omega)|=\frac{|a_o|}{\sqrt{u^2+B}}. \quad \text{(A3)}$$

Squaring Eq. (A3) produces Eq. (9), as required.

As argued earlier, the maximum gain is given by Eq. (10). Hence,

$$\left|T\left(\omega\right)\right|_{\max}=\frac{\left|a_o\right|}{\sqrt{\omega_p{}^4-\left(1-\dfrac{1}{2Q^2}\right)^2\omega_p{}^4}}$$

$$\begin{aligned}\left|T\left(\omega\right)\right|_{\max}&=\frac{\left|a_0\right|}{\omega_p{}^2\sqrt{\left[1-\left(1-\dfrac{1}{2Q^2}\right)^2\right]}}\\&=\frac{\left|a_0\right|}{\omega_p{}^2\sqrt{\left[1-\left[1-\dfrac{1}{Q^2}+\dfrac{1}{4Q^4}\right]\right]}}\\&=\frac{\left|a_0\right|}{\omega_p{}^2\sqrt{\dfrac{1}{Q^2}-\dfrac{1}{4Q^4}}}\\&=\frac{\left|a_0\right|}{\omega_p{}^2\sqrt{\dfrac{1}{Q^2}\left(1-\dfrac{1}{4Q^2}\right)}}\\&=\frac{\left|a_0\right|Q}{\omega_p{}^2\sqrt{1-\dfrac{1}{4Q^2}}}.\end{aligned}\qquad\text{(A4)}$$

Clearly, Eq. (A4) is the same as Eq. (12), as required.

APPENDIX B

In this appendix, we show that the square of the gain of the second-order high-pass filter is given by Eq. (9), and the maximum gain is given by Eq. (14).

Recall that the gain of a second order high-pass filter is given by Eq. (4). Expanding the brackets of this equation produces

$$\left|T(\omega)\right|=\frac{\left|a_2\right|\omega^2}{\sqrt{\omega_p{}^4-2\omega_p{}^2\omega^2+\omega^4+\dfrac{\omega^2\omega_p{}^2}{Q^2}}}.\qquad\text{(B1)}$$

Dividing the numerator and denominator of (B1) by ω^2 gives

$$\begin{aligned}\left|T(\omega)\right|&=\frac{\left|a_2\right|}{\sqrt{\left[\left(\dfrac{\omega_p}{\omega}\right)^4-2\left(\dfrac{\omega_p}{\omega}\right)^2+1+\left(\dfrac{\omega_p}{\omega}\right)^2\dfrac{1}{Q^2}\right]}}\\&=\frac{\left|a_2\right|}{\sqrt{\left[\left(\dfrac{\omega_p}{\omega}\right)^4+\left(\dfrac{\omega_p}{\omega}\right)^2\left(-2+\dfrac{1}{Q^2}\right)+1\right]}}.\end{aligned}\qquad\text{(B2)}$$

Completing the square in the denominator of Eq. (B2) gives

$$\left|T(\omega)\right|=\frac{\left|a_2\right|}{\sqrt{\left[\left(\dfrac{\omega_p}{\omega}\right)^2-\left(1-\dfrac{1}{2Q^2}\right)\right]^2+1-\left(1-\dfrac{1}{2Q^2}\right)^2}}.\qquad\text{(B3)}$$

Hence, Eq. (B3) can be rewritten as

$$\left|T(\omega)\right|=\frac{\left|a_2\right|}{\sqrt{u^2+B}},\qquad\text{(B4)}$$

where $u=\left(\dfrac{\omega_p}{\omega}\right)^2-\left(1-\dfrac{1}{2Q^2}\right)$ and $B=1-\left(1-\dfrac{1}{2Q^2}\right)^2$.

Squaring Eq. (B4) gives Eq. (9).

As determined earlier, the maximum gain is given by Eq. (10). Hence,

$$\begin{aligned}\left|T\left(\omega\right)\right|_{\max}&=\frac{\left|a_2\right|}{\sqrt{B}}\\&=\frac{\left|a_2\right|}{\sqrt{\left[1-\left[1-\dfrac{1}{Q^2}+\dfrac{1}{4Q^4}\right]\right]}}\\&=\frac{\left|a_2\right|}{\sqrt{\dfrac{1}{Q^2}\left(1-\dfrac{1}{4Q^2}\right)}}=\frac{\left|a_2\right|Q}{\sqrt{1-\dfrac{1}{4Q^2}}}.\end{aligned}\qquad\text{(B5)}$$

Clearly, Eq. (B5) is the same as Eq. (14), as required.

APPENDIX C

In this appendix we show that the square of the gain of the second-order band-pass filter is given by Eq. (9) and the maximum gain is given by Eq. (16).

Recall that the gain of a second order band-pass filter is given by Eq. (6). Dividing its numerator and denominator by ω gives

$$|T(\omega)| = \frac{|a_1|}{\sqrt{\left(\frac{\omega_p^2-\omega^2}{\omega}\right)^2+\left(\frac{\omega_p}{Q}\right)^2}}. \quad \text{(C1)}$$

Clearly, Eq. (C1) can be written as

$$|T(\omega)| = \frac{|a_1|}{\sqrt{u^2+B}}, \quad \text{(C2)}$$

if $u = \frac{\omega_p^2-\omega^2}{\omega}$ and $B = \left(\frac{\omega_p}{Q}\right)^2$. Squaring Eq. (C2) produces Eq. (9).

As mentioned earlier, the maximum gain is given by Eq. (10). Hence,

$$\left|T(\omega)\right|_{\max} = \frac{|a_1|}{\sqrt{B}} = \frac{|a_1|}{\sqrt{\frac{\omega_p^2}{Q^2}}} = \frac{|a_1|Q}{\omega_p}. \quad \text{(C3)}$$

APPENDIX D

In this appendix, we show that the square of the gain of the second-order low-pass notch and high-pass notch filters is given by Eq. (9). Furthermore, we show that the frequency at which the maximum gain occurs is given by Eq. (17) and the maximum gain itself is given by Eq. (18).

Recall that the gain for this filter is given by Eq. (8). Squaring this gives:

$$|T(\omega)|^2 = \frac{|a_2|^2\left[\omega_z^{\,2}-\omega^2\right]^2}{\left(\omega_p^2-\omega^2\right)^2+\left(\frac{\omega\omega_p}{Q}\right)^2}. \quad \text{(D1)}$$

Diving the numerator and denominator of (D1) by $\omega_p^{\;4}$ and simplifying gives:

$$|T(\omega)|^2 = \frac{|a_2|^2\left[\left(\frac{\omega_z}{\omega_p}\right)^2-\left(\frac{\omega}{\omega_p}\right)^2\right]^2}{\left[1-\left(\frac{\omega}{\omega_p}\right)^2\right]^2+\left(\frac{\omega}{\omega_p}\right)^2\left(\frac{1}{Q}\right)^2}. \quad \text{(D2)}$$

Let $k = \left(\frac{\omega_z}{\omega_p}\right)^2$ and $y = k-\left(\frac{\omega}{\omega_p}\right)^2$. Hence,

$$\left(\frac{\omega}{\omega_p}\right)^2 = k-y. \quad \text{(D3)}$$

Placing (D3) into (D2) and simplifying gives:

$$\left|T(\omega)\right|^2 = \frac{|a_2|^2\, y^2}{(1-k+y)^2+\frac{k-y}{Q^2}}$$

$$= \frac{|a_2|^2\, y^2}{y^2+2[1-k]y+(1-k)^2+\frac{k}{Q^2}-\frac{y}{Q^2}}$$

$$\left|T(\omega)\right|^2 = \frac{|a_2|^2\, y^2}{y^2+2\left[1-k-\frac{1}{2Q^2}\right]y+(1-k)^2+\frac{k}{Q^2}}. \quad \text{(D4)}$$

Equation (D4) can be rewritten as

$$|T(\omega)|^2 = \frac{|a_2|^2}{1+2C\frac{1}{y}+D\frac{1}{y^2}}, \quad \text{(D5)}$$

where $C = 1-k-\frac{1}{2Q^2}$ and $D = (1-k)^2+\frac{k}{Q^2}$.

However, by completion of squares, Eq. (D5) becomes

$$|T(\omega)|^2 = \frac{|a_2|^2}{\left(\frac{\sqrt{D}}{y}+\frac{C}{\sqrt{D}}\right)^2+1-\frac{C^2}{D}}. \quad \text{(D6)}$$

Hence, Eq. (D6) is indeed equal to Eq. (9) if $u = \frac{\sqrt{D}}{y}+\frac{C}{\sqrt{D}}$, and $B = 1-\frac{C^2}{D}$.

Recall that Eq. (D6) is a maximum if $=0, i.e., \frac{\sqrt{D}}{y_{max}}+\frac{C}{\sqrt{D}} = 0$. Hence, $y_{max} = -\frac{D}{C}$, or

$$y_{\max} = k-\left(\frac{\omega_{\max}}{\omega_p}\right)^2 = -\frac{(1-k)^2+\frac{k}{Q^2}}{1-k-\frac{1}{2Q^2}}. \quad \text{(D7)}$$

Solving for $\omega_{\max}$ in Eq. (D7) yields:

$$\omega_{\max} = \omega_p\sqrt{k+\frac{(1-k)^2+\frac{k}{Q^2}}{1-k-\frac{1}{2Q^2}}}. \quad \text{(D8)}$$

Simplifying Eq. (D8) gives:

$$\omega_{\max}=\omega_p\sqrt{\frac{k\left(1-k-\frac{1}{2Q^2}\right)+(1-k)^2+\frac{k}{Q^2}}{1-k-\frac{1}{2Q^2}}}$$

$$\omega_{\max}=\omega_p\sqrt{\frac{k-k^2-\frac{k}{2Q^2}+1-2k+k^2+\frac{k}{Q^2}}{1-k-\frac{1}{2Q^2}}}$$

$$=\omega_p\sqrt{\frac{\frac{k}{2Q^2}+1-k}{1-k-\frac{1}{2Q^2}}}. \tag{D9}$$

Multiplying the numerator and denominator inside the radical of Eq. (D9) by −1 gives

$$\omega_{\max}=\omega_p\sqrt{\frac{k-\frac{k}{2Q^2}-1}{k+\frac{1}{2Q^2}-1}}.$$

Hence,

$$\omega_{\max}=\omega_p\sqrt{\frac{\left(\frac{\omega_z}{\omega_p}\right)^2\left[1-\frac{1}{2Q^2}\right]-1}{\left(\frac{\omega_z}{\omega_p}\right)^2+\frac{1}{2Q^2}-1}}. \tag{D10}$$

The reader should recognize Eq. (D10) as Eq. (17), which is what we wanted to show.

Now recall that the maximum gain is given by Eq. (10). Squaring this gives

$$\left|T(\omega)\right|^2_{\max}=\frac{|a_2|^2}{1-\frac{C^2}{D}}$$

$$\left|T(\omega)\right|^2_{\max}=\frac{|a_2|^2}{1-\frac{\left(1-k-\frac{1}{2Q^2}\right)^2}{(1-k)^2+\frac{k}{Q^2}}}. \tag{D11}$$

Equation (D11) can be rewritten as

$$\left|T(\omega)\right|^2_{\max}=\frac{|a_2|^2}{\frac{(1-k)^2+\frac{k}{Q^2}-\left(1-k-\frac{1}{2Q^2}\right)^2}{(1-k)^2+\frac{k}{Q^2}}}$$

$$=\frac{|a_2|^2}{\frac{(1-k)^2+\frac{k}{Q^2}-\left((1-k)^2-\frac{1-k}{Q^2}+\frac{1}{4Q^2}\right)}{(1-k)^2+\frac{k}{Q^2}}}. \tag{D12}$$

Hence, Eq. (D12) becomes

$$\left|T(\omega)\right|^2_{\max}=\frac{|a_2|^2\left((1-k)^2+\frac{k}{Q^2}\right)}{\frac{1}{Q^2}\left(1-\frac{1}{4Q^2}\right)}. \tag{D13}$$

Taking the square root of Eq. (D13) gives Eq. (18).