Maximizing the height or flight time for a vertical launch of fixed energy

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Abstract

A standard problem in introductory physics is to determine the peak height and flight time of a ball projected at a specified initial *speed* without drag. Here a ball is instead launched vertically upward in the presence of quadratic air resistance with a fixed initial *kinetic energy* so that its launch speed depends on the mass of the ball, as does the quadratic drag force. The optimal radius of ball to reach either the greatest peak height or flight time is calculated.

Keywords: Projectile motion, quadratic drag, mechanical energy.

Resumen

Un problema estándar en física introductoria es determinar la altura máxima y el tiempo de vuelo de una pelota proyectada a una velocidad inicial especificada sin arrastre. Aquí, en cambio, se lanza una pelota verticalmente hacia arriba en presencia de una resistencia cuadrática del aire con una energía cinética inicial fija, de modo que su velocidad de lanzamiento depende de la masa de la pelota, al igual que la fuerza de arrastre cuadrática. Se calcula el radio óptimo de la bola para alcanzar la mayor altura máxima o el tiempo de vuelo.

Palabras clave: Movimiento de proyectil, arrastre cuadrático, energía mecánica.

I. INTRODUCTION

If a ball is launched with a specified kinetic energy, its initial speed depends on the mass of the ball. A bowling ball has a large mass and thus a small launch speed, but it is only slightly affected by air resistance. A ping-pong ball would be launched at a far higher speed to have the same kinetic energy, but it also will only reach a small height because air drag takes a large toll. This reasoning suggests that for any given launch energy, there is an optimal ball mass (or radius) to reach the greatest height. One might likewise expect a (different) optimal mass to maximize the total flight time of the ball up and back down.

II. MOTION UNDER GRAVITY WITH QUADRATIC DRAG

Consider a ball of mass m launched vertically upward (defining the $+y$ axis) in a uniform downward gravitational field of magnitude g, neglecting buoyancy and the added mass term [1]. The decrease in its speed υ with time t as it rises upward is described by Newton's second law as

$$m\frac{d\upsilon}{dt} = -mg - \frac{1}{2}C_\mathrm{D}\rho_\mathrm{air}A\upsilon^2 \tag{1}$$

where C_D is the drag coefficient, A is the cross-sectional area of the ball, and ρ_air is the atmospheric density [2]. The chain rule implies that

$$m\frac{d\upsilon}{dt} = m\frac{dy}{dt}\frac{d\upsilon}{dy} = m\upsilon\frac{d\upsilon}{dy} = \frac{d}{dy}\left(\frac{1}{2}m\upsilon^2\right) \tag{2}$$

so that Eq. (1) becomes

$$\frac{dK}{dy} = -mg - \frac{C_\mathrm{D}\rho_\mathrm{air}A}{m}K \tag{3}$$

where $K = \frac{1}{2}m\upsilon^2$ is the kinetic energy of the ball. Equation (3) integrates as

$$K = \left(K_0 + E\right)\exp\left(-mgy/E\right) - E \tag{4}$$

where K_0 is the (fixed) launch kinetic energy at $y = 0$, and a characteristic energy is

$$E \equiv \frac{m^2 g}{C_\mathrm{D}\rho_\mathrm{air}A} = \frac{16\pi g\rho_\mathrm{ball}^2}{9C_\mathrm{D}\rho_\mathrm{air}}R^4 \equiv BR^4 \tag{5}$$

in terms of the radius R and average density ρ_ball of the ball. Here B is a constant with units of J/m^4. The ball attains its peak height $y = H$ when $K = 0$ so that

$$H = \frac{4\rho_{\text{ball}} R}{3C_D \rho_{\text{air}}} \ln\left(1 + \frac{K_0}{BR^4}\right) \tag{6}$$

according to Eqs. (4) and (5). Introducing the dimensionless energy ratio

$$X \equiv \frac{K_0}{E} = \frac{3C_D \rho_{\text{air}} \upsilon_0^2}{8g\rho_{\text{ball}} R} \tag{7}$$

where υ_0 is the launch speed of the ball, Eq. (6) becomes

$$H = \frac{\upsilon_0^2}{2gX} \ln(1+X). \tag{8}$$

As a check, the drag force in Eq. (1) becomes zero in the limit as $C_D \to 0$, in which case $X \to 0$ according to Eq. (7). But $\ln(1+X) \to X$ in that limit, so that Eq. (8) reduces to the familiar drag-free result $H_0 = \upsilon_0^2 / 2g$. Equation (8) is graphed in Fig. 1. As drag increases, X increases and the peak height H correspondingly decreases.

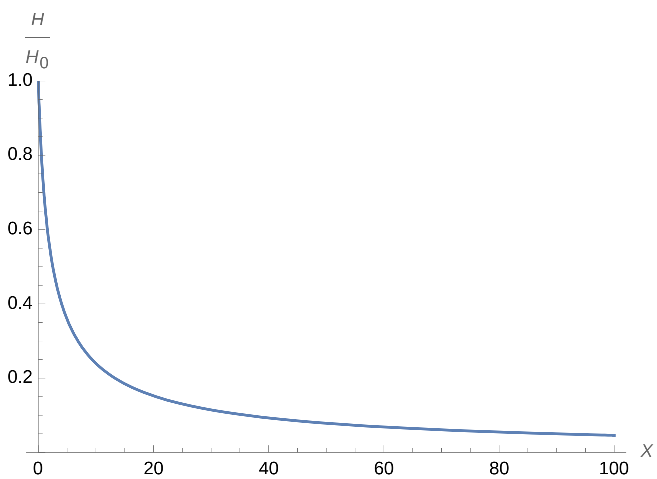

FIGURE 1. Ratio of peak heights in the presence and absence of air drag as a function of the dimensionless ratio of energies defined in Eq. (7).

III. MAXIMIZING THE PEAK HEIGHT

In the presence of drag, the peak height is optimized by setting $dH/dR = 0$ in Eq. (6) to obtain

$$\ln\left(1 + K_0 / BR^4\right) = 4\frac{K_0 / BR^4}{1 + K_0 / BR^4}. \tag{9}$$

This equation can be rewritten as

$$X = \exp\left[4 \middle/ \left(1 + X^{-1}\right)\right] - 1 \tag{10}$$

which can be solved by iteration. A starting value is obtained by assuming X is large to obtain

$$X \approx \exp[4] - 1 \approx 54. \tag{11}$$

Substitute this value back into the right-hand side of Eq. (10) to obtain the improved value $X \approx 50$ (which can be further iterated if increased accuracy is desired, but is already within about 1% of the exact value of $X_{\text{opt}} \approx 49.435$ that can be obtained using the Lambert *W* function [3]). This value can be substituted into Eqs. (7) and (8) to find the optimized radius and peak height, respectively, of the ball. Specifically,

$$R_{\text{opt}} = \left(\frac{9C_D \rho_{\text{air}} K_0}{16\pi g \rho_{\text{ball}}^2 X_{\text{opt}}}\right)^{1/4} \tag{12}$$

for which $H_{\text{opt}} / H_0 \approx 7.93\%$ in accord with Fig. 1.

IV. MAXIMIZING THE FLIGHT TIME

Equation (1) for the upward motion of the ball can be rewritten as

$$g^{-1} \frac{d\upsilon}{dt} = -1 - X \frac{\upsilon^2}{\upsilon_0^2} \tag{13}$$

using Eq. (7). Equation (13) integrates as

$$\upsilon = \frac{\upsilon_0}{\sqrt{X}} \tan\left(\tan^{-1}\sqrt{X} - \frac{gt}{\upsilon_0}\sqrt{X}\right) \tag{14}$$

for the initial condition $\upsilon = \upsilon_0$ at $t = 0$. If the drag is weak, such that $X \ll 1$, then Eq. (14) reduces to the familiar kinematic equation

$$\upsilon \approx \frac{\upsilon_0}{\sqrt{X}}\left(\sqrt{X} - \frac{gt}{\upsilon_0}\sqrt{X}\right) = \upsilon_0 - gt \tag{15}$$

which decreases linearly with time. On the other hand, when the drag is strong, the ball's velocity decreases nonlinearly. For example, when $X = X_{\text{opt}} \approx 49.435$ as found after Eq. (11), a graph of Eq. (14) is presented in Fig. 2 for the range of values of υ/υ_0 decreasing from 1 to 0.

The ball reaches its maximum height when $\upsilon = 0$ at a time of

$$t_{\text{up}} = \frac{\upsilon_0}{g\sqrt{X}} \tan^{-1}\sqrt{X} \tag{16}$$

according to Eq. (14). In the drag-free limit as $X \to 0$, this upward flight time becomes $t_0 = \upsilon_0 / g$ as expected.

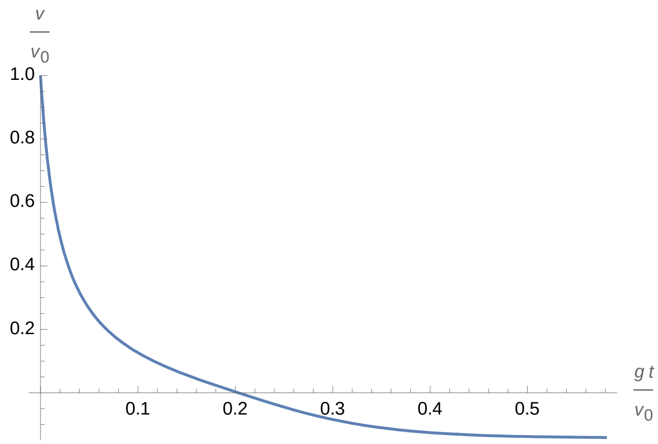

FIGURE 2. Ratio of the velocity of the ball to its launch velocity as a function of the dimensionless time after launch for $X = X_{opt}$.

An analytical solution for the ascent of the ball as a function of time can be obtained by replacing υ with dy/dt in Eq. (14) and integrating to get

$$y = \frac{\upsilon_0^2}{gX}\ln\left[\sqrt{1+X}\,\cos\left(\tan^{-1}\sqrt{X} - \frac{gt}{\upsilon_0}\sqrt{X}\right)\right] \quad (17)$$

for the initial condition $y = 0$ at $t = 0$. In the drag-free limit as $X \to 0$, the cosine term is approximately 1 and the square root term in front of it is approximately $1 + X/2$, so that $y_0 = \upsilon_0^2 / 2g$ as expected. Equation (17) scaled by the peak height H from Eq. (8) is graphed in the rising portion of Fig. 3 for the same value of $X = X_{opt} \approx 49.435$ as used in Fig. 2.

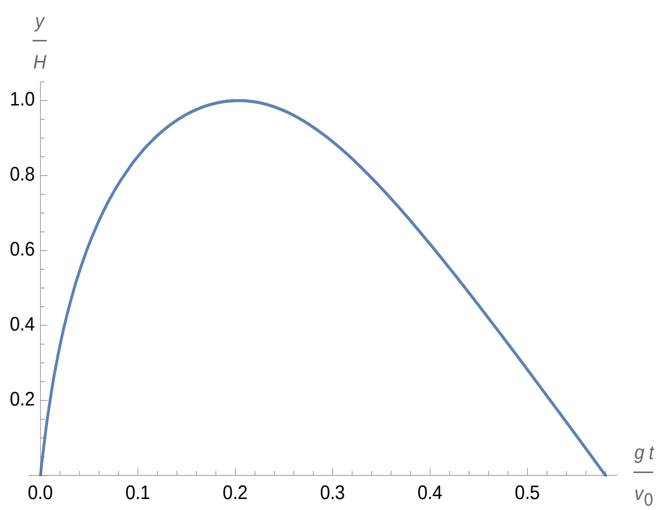

FIGURE 3. Ratio of the height of the ball to its peak height as a function of the dimensionless time after launch for $X = X_{opt}$.

For the downward motion of the ball back to its launch point, it is convenient to redefine the y-axis with the origin at the maximum height and downward positive, so that Eq. (1) becomes

$$m\frac{d\upsilon}{dt} = mg - \frac{1}{2}C_D\rho_{air}A\upsilon^2 \quad (18)$$

and thus Eq. (13) becomes

$$g^{-1}\frac{d\upsilon}{dt} = 1 - X\frac{\upsilon^2}{\upsilon_0^2} \quad (19)$$

where υ_0 remains the launch speed. Restarting the timer so that $t = 0$ refers to the topmost point of the ball's trajectory where $\upsilon = 0$, Eq. (19) integrates as

$$\upsilon = \frac{\upsilon_0}{\sqrt{X}}\tanh\left(\frac{gt}{\upsilon_0}\sqrt{X}\right). \quad (20)$$

If the drag is weak, such that $X \ll 1$, then Eq. (20) becomes $\upsilon \approx gt$ which increases linearly with time as expected. On the other hand, in the presence of significant drag, the ball's speed asymptotically approaches the terminal value $\upsilon_T = \upsilon_0/\sqrt{X}$ if it falls far enough [4]. The negative of Eq. (20) shifted horizontally to the right is graphed in Fig. 2 over the vertical range of values from 0 to $-\upsilon_T/\upsilon_0$ for $X = X_{opt} \approx 49.435$.

Replacing υ with dy/dt in Eq. (20) and integrating, the descent of the ball as a function of time is described by

$$y = \frac{\upsilon_0^2}{gX}\ln\cosh\left(\frac{gt}{\upsilon_0}\sqrt{X}\right) \quad (21)$$

for the initial condition $y = 0$ at $t = 0$. The descending portion of the graph in Fig. 3 plots this expression ratioed to H, flipped over vertically, and shifted horizontally to the right for $X = X_{opt} \approx 49.435$. Equating Eqs. (8) and (21), the downward flight time of the ball back to its launch point is

$$t_{down} = \frac{\upsilon_0}{g\sqrt{X}}\cosh^{-1}\sqrt{1+X} \quad (22)$$

which again reduces to $t_0 = \upsilon_0 / g$ in the drag-free limit as $X \to 0$. The total up and down flight time is obtained by adding together Eqs. (16) and (22) to get

$$T = \frac{\upsilon_0}{g\sqrt{X}}\left(\tan^{-1}\sqrt{X} + \cosh^{-1}\sqrt{1+X}\right) \quad (23)$$

where the total flight time is $T_0 = 2\upsilon_0 / g$ in the absence of drag. Their ratio is plotted in Fig. 4, showing that the flight time *decreases* monotonically with increasing drag. Although the ball's speed (at any given height after launch) is smaller in the presence of drag than in its absence, the ball subject to drag also climbs up to a smaller peak height. The second effect dominates over the first effect for quadratic drag, so that in a race between two balls simultaneously launched upward with the same starting speed, one inside an evacuated tube and one in air, the one in air would return to the ground first [5].

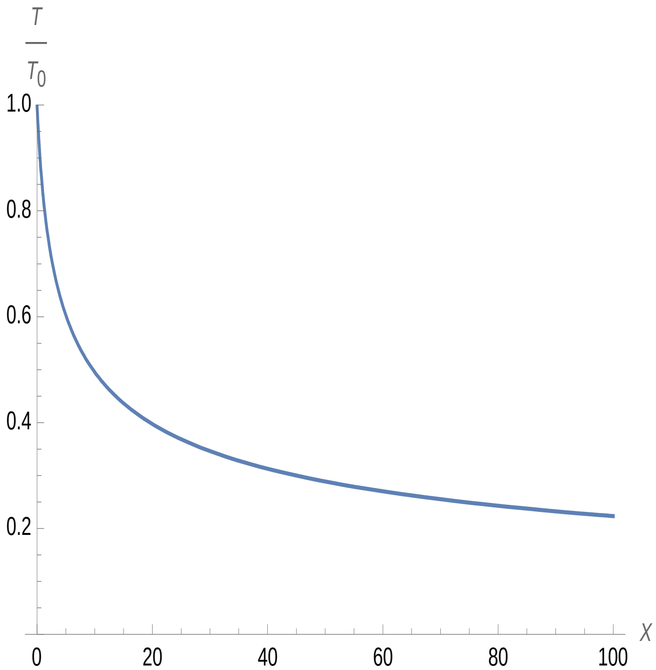

FIGURE 4. Ratio of flight times in the presence and absence of air drag as a function of the dimensionless ratio of energies defined in Eq. (7).

Using Eqs. (5), (7), and (23), the total flight time is written in terms of the radius of the ball as

$$T = \sqrt{\frac{8\rho_{\text{ball}}R}{3gC_D\rho_{\text{air}}}}\left(\tan^{-1}\sqrt{\frac{K_0}{BR^4}} + \cosh^{-1}\sqrt{1+\frac{K_0}{BR^4}}\right) \quad (24)$$

which is optimized by setting $dT/dR = 0$ to obtain

$$\tan^{-1}\sqrt{X} + \cosh^{-1}\sqrt{1+X} = 4\sqrt{X}\left[(1+X)^{-1} + (1+X)^{-1/2}\right] \quad (25)$$

whose numerical solution is $X'_{\text{opt}} \approx 88.0$ which can be substituted into Eq. (12) to find the optimal radius with corresponding flight time $T_{\text{opt}}/T_0 \approx 23.5\%$ in agreement with Fig. 4.

V. APPLICATIONS

Suppose one uses a spring-loaded ball launcher with 100% conversion efficiency from elastic energy into kinetic energy so that $K_0 = \frac{1}{2}kx^2$ where k is the spring constant and x is the initial compression of the spring. Then Eq. (12) predicts the radius of ball that will achieve the largest peak height for this launcher.

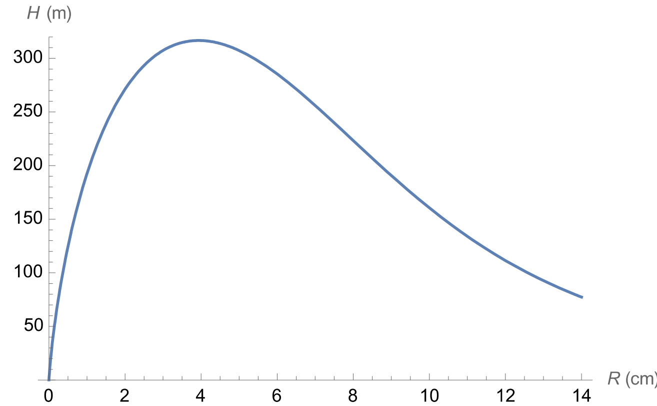

FIGURE 5. Peak height attained by the ball as a function of its radius for the parameters specified in the text.

For example, suppose the drag coefficient is $C_D = 0.5$, the air density is $\rho_{\text{air}} = 1.3$ kg/m^3, the average density of the ball is $\rho_{\text{ball}} = 1000$ kg/m^3, the gravitational field strength is $g = 9.8$ m/s^2, and the launcher imparts an initial kinetic energy of $K_0 = 10$ kJ (using say a spring with $k = 20$ kN/m and $x = 1$ m). Then Fig. 5 plots the peak height of the ball as a function of its radius from Eqs. (5) and (6). The maximum is $H_{\text{opt}} = 320$ m when $R_{\text{opt}} = 3.9$ cm.

If a ship were to launch a distress flare, one might wish to instead maximize its flight time for a given launch energy, so as to give it the longest chance of being seen. Using the same parameters as above, Fig. 6 graphs the flight time as a function of the radius. The maximum is $T_{\text{opt}} = 17$ s at a slightly different radius (than in Fig. 5) of $R_{\text{opt}} = 3.4$ cm, according to the final paragraph of Sec. IV.

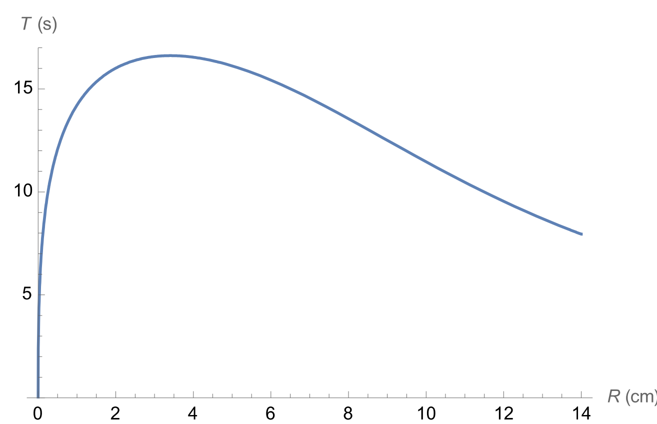

FIGURE 6. Total flight time of the ball as a function of its radius for the parameters specified in the text.

VI. CONCLUSION

This article has explored the properties of spheres launched vertically with a given initial kinetic energy. Numerical solution determines the optimal ball radius (or mass) to reach the greatest height or flight time, depending on the strength of the quadratic air drag.

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