Projectile Motion from Free Fall

Yuji Kajiyama
Faculty of Education, Gifu Shotoku Gakuen University, Takakuwa-Nishi 1-1, Yanaizu, Gifu, 501-6194, Japan.

E-mail: kajiyama@gifu.shotoku.ac.jp

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Abstract

The range of a projectile without air resistance depends on the launch angle, the initial speed, and the launch height. If the speed of a free fall at a certain height directly reflects the initial speed of a projectile thrown from that height, the launch height is a key factor because a larger initial speed requires a lower launch height. In this article, we discuss the maximum range of a projectile launched with an initial speed caused by free fall. This problem is suitable for undergraduate students in calculus-based physics courses.

Keywords: Classical Mechanics, Projectile Motion, Optimization, Physics Education.

Resumen

El alcance de un proyectil sin resistencia del aire depende del ángulo de lanzamiento, la velocidad inicial y la altura de lanzamiento. Si la velocidad de caída libre a cierta altura refleja directamente la velocidad inicial de un proyectil lanzado desde esa altura, la altura de lanzamiento es un factor clave, ya que una mayor velocidad inicial requiere una menor altura de lanzamiento. En este artículo, analizamos el alcance máximo de un proyectil lanzado con una velocidad inicial causada por la caída libre. Este problema es adecuado para estudiantes de grado en física con base en cálculo.

Palabras clave: Mecánica clásica, Movimiento de proyectiles, Optimización, Educación en Física.

I. INTRODUCTION

The projectile motion on the ground is a topic that students are sure to study in introductory physics courses. When a projectile is launched from and lands on the ground, the horizontal range of the projectile is maximized at a 45° angle to the horizontal. However, if the projectile is launched, with the same initial speed, at a certain height above the ground, then the optimal angle is less than 45°. This results in an increased maximum range because of the initial height [1, 2, 3, 4, 5].

What happens if the initial speed depends on the initial height? If the speed of free fall at a height h directly reflects the initial speed of the projectile thrown from that height h, the larger the initial speed, the lower the initial position of the projectile. In that case, what condition would result in the maximum range? In this short article, we discuss the maximum range problem in the following two cases: (i) to find the optimal angle when the launch height h is given, (ii) to find the optimal height when the initial launch angle θ is given. The case (i) can be done without calculus, while the case (ii) requires solving an extremum problem with differentiation, finding that the behavior changes completely at an angle of 30°. This optimization problem is suitable for undergraduate students in calculus-based physics courses.

II. PROJECTILE MOTION

Consider an object falling freely from a starting point at a height H. It then bounds off a board placed at a height h, changing its direction at an angle θ to the horizontal as shown in FIGURE 1. After bounding off the board perfectly elastically, the object moves as a projectile with the initial speed

$$v_h = \sqrt{2gH(1-\eta)}, \qquad \eta = \frac{h}{H}, \tag{1}$$

where g is the gravitational acceleration.

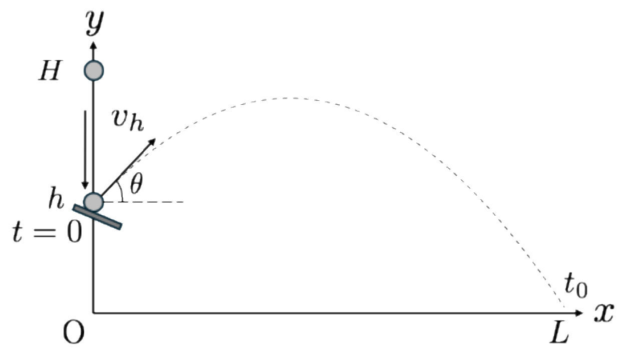

FIGURE 1. Schematic diagram of free fall and its subsequent projectile motion.

The object finally lands at the point of the range L. The problem is to find the condition to maximize the range L for various situations.

After performing some basic calculations, one finds the position of the projectile after being launched from the point $(0, h)$ at $t = 0$ as

$$x(t) = v_h t \cos\theta, \tag{2}$$

$$y(t) = -\frac{1}{2}gt^2 + v_h t \sin\theta + h. \tag{3}$$

The object lands at the point $(L, 0)$ at time t_0. The condition $y(t_0) = 0$ gives

$$t_0 = \frac{v_h \sin\theta}{g}(1 + z), \tag{4}$$

and thus the range $L = x(t_0)$ is

$$L = \frac{v_h^2 \sin 2\theta}{2g}(1 + z), \tag{5}$$

where

$$z = \sqrt{1 + \frac{1}{\sin^2\theta}\left(\frac{2gH}{v_h^2} - 1\right)}. \tag{6}$$

Since $0 \le v_h^2 \le 2gH$ from Eq. (1), z is real and positive definite.

We will discuss the maximum value of L in the following two cases: (i) to find the optimal angle $\theta_{\max}$ when h is given, and (ii) to find the optimal height $h_{\max}$ when θ is given.

(i) $\boldsymbol{\theta_{\max}}$ for a given $\boldsymbol{h}$

Suppose that the board is placed at a point of height h, and its angle can be changed freely. The problem is to find the optimal angle $\theta_{\max}$ to maximize the range L as a function of h.

Following Ref. [3], $L^2 = x(t_0)^2$ can be written as

$$\begin{aligned} L^2 &= v_h^2 t_0^2 \cos^2\theta, \\ &= v_h^2 t_0^2 - \left(\frac{1}{2}gt_0^2 - h\right)^2, \\ &= -\frac{1}{4}g^2\left[t_0^2 - \frac{2H(2-\eta)}{g}\right]^2 + 4H^2(1-\eta). \end{aligned} \tag{7}$$

The maximum range $L_{\max}$ requires the landing time $t_{0\max} = \sqrt{2H(2-\eta)/g}$, and thus we obtain $L_{\max}$ and its corresponding launch angle $\theta_{\max}$ as a function of η, as

$$\tilde{L}_{\max} = \frac{L_{\max}}{H} = 2\sqrt{1-\eta}, \tag{8}$$

and

$$\tan\theta_{\max} = \frac{\frac{1}{2}gt_{0\max}^2 - h}{L_{\max}} = \sqrt{1-\eta}. \tag{9}$$

Therefore, when the board is placed at a height h, the maximum range is given by Eq. (8), provided that the launch angle is given by Eq. (9).

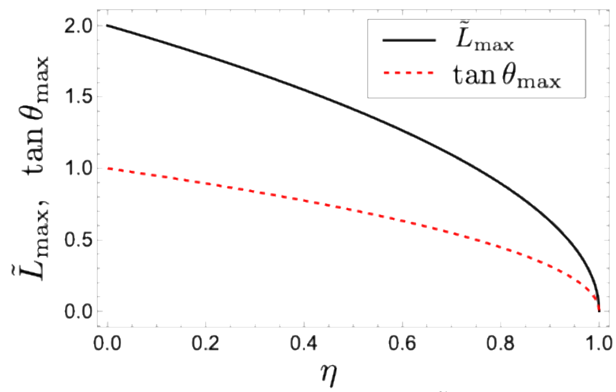

FIGURE 2. The $\eta = h/H$ dependence of $\tilde{L}_{\max} = L_{\max}/H$ (black) and $\tan\theta_{\max}$ (red dashed).

Fig. 2 shows the η dependence of $\tilde{L}_{\max}$ (black) and $\tan\theta_{\max}$ (red dashed) drawn from Eqs. (8) and (9), respectively. As seen from the figure, as η increases, $\tilde{L}_{\max}$ decreases. When $\eta = 0$, $\tan\theta_{\max} = 1$ and $\tilde{L}_{\max} = 2$ corresponding to the projectile motion from the ground with the initial speed of $\sqrt{2gH}$ and the angle of 45°.

(ii) $\boldsymbol{h_{\max}}$ for a given $\boldsymbol{\theta}$

Next, suppose that the angle of the board is fixed, and that the launch angle is θ. What is the optimal height of the board $h_{\max}$ that maximizes the range L ?

We first examine the v_h-dependence of L of Eq. (5). The first derivative of L with respect to v_h^2 is

$$\frac{dL}{d(v_h^2)} = \frac{\sin 2\theta}{4gz}\left(z + 1 + \frac{1}{\sin\theta}\right)\left(z + 1 - \frac{1}{\sin\theta}\right). \tag{10}$$

Since $z \ge 0$ and $\sin 2\theta \ge 0$ for $0 \le \theta \le 90°$, the sign of Eq. (10) depends on that of $f(v_h) = z + 1 - 1/\sin\theta$. The values of $f(v_h)$ for the lower and upper limits of v_h,

$$\begin{aligned} f(v_h) &\to \infty && \text{as} \quad v_h \to 0, \\ f(v_h) &\to 2 - \frac{1}{\sin\theta} && \text{as} \quad v_h \to \sqrt{2gH}. \end{aligned}$$

Therefore, if $\sin\theta < 1/2$ $(\theta < 30°)$, the sign of $f(v_h)$ changes at a certain value of v_h. This indicates that for $\theta < 30°$, the range L has a maximum at a certain value of v_h. Conversely, for $\theta \ge 30°$, L is a monotonically increasing function of v_h, that is, a monotonically decreasing function of h.

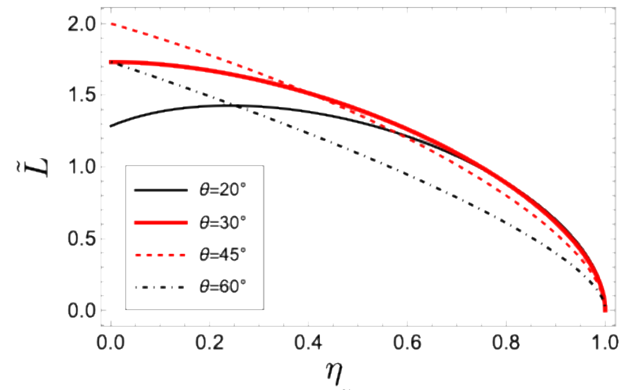

FIGURE 3. The η dependence of $\tilde{L} = L/H$ from Eqs. (5), (6), and (1) for various values of θ. The black solid curve for $\theta = 20°$ has a maximum.

Fig. 3 shows the η dependence of $\tilde{L} = L/H$ drawn from Eqs. (5), (6), and (1) for various values of θ. As one can see that $\tilde{L}$ is monotonically decreasing function of η except for $\theta = 20°$ (black solid curve). The maximum of the function for $\theta = 20°$ is at about $\eta = 0.3$, while the red solid curve for $\theta = 30°$ has the maximum at $\eta = 0$. Therefore, for $\theta \geq 30°$, the maximum range $\tilde{L}_{\max}$ is given by Eq. (5) when $v_h = \sqrt{2gH}$ $(\eta = 0)$ as

$$\theta \geq 30° : \; \tilde{L}_{\max} = 2\sin 2\theta. \tag{11}$$

Then, what is the condition of the maximum of L for $\theta < 30°$? The extremum condition from Eq. (10) is

$$z + 1 - \frac{1}{\sin\theta} = 0, \tag{12}$$

that gives

$$v_{h\max} = \sqrt{\frac{gH}{1 - \sin\theta}}, \tag{13}$$

or

$$\eta_{\max} = \frac{1 - 2\sin\theta}{2(1 - \sin\theta)}. \tag{14}$$

Therefore, the maximum range for $\theta < 30°$ is given by

$$\theta < 30° : \; \tilde{L}_{\max} = \frac{\cos\theta}{1 - \sin\theta}. \tag{15}$$

Fig. 4 shows the θ dependence of $\eta_{\max}$ (red dashed) and $\tilde{L}_{\max}$ (black solid). For $\theta < 30°$, these are drawn from Eqs. (14) and (15), and for $\theta \geq 30°$ from Eq. (11) and $\eta_{\max} = 0$. When the launch angle θ is fixed in the region of $0 \leq \theta < 30°$, one must place the board at a certain height given in Eq. (14) to maximize the range L. When the launch angle is $\theta \geq 30°$, the board should be placed on the ground.

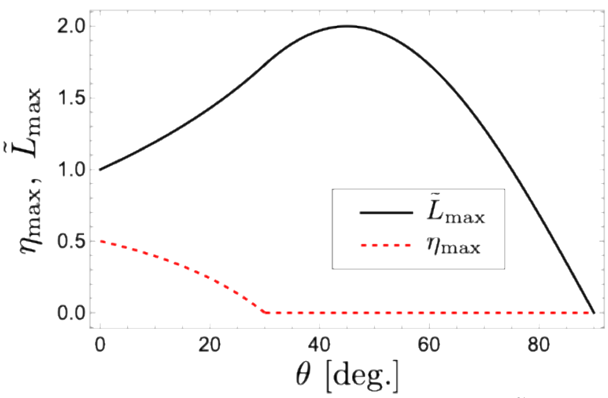

FIGURE 4. The θ dependence of $\eta_{\max}$ and $\tilde{L}_{\max}$.

III. CONCLUSIONS

When an object is dropped freely from a height H and then bounds at an angle off a board at a height h, it performs a projectile motion. The initial speed depends on the height of the board h. In such a case, it is not obvious where to place the board and at what launch angle to maximize the range. In this paper, we determined the conditions for the maximum range, where the launch angle is fixed, and where the board height is fixed. In the former case, the solution can be obtained without differentiation. We found that the maximum range and the tangent of the corresponding angle are proportional to $\sqrt{1 - h/H}$. In the latter case, by solving the extremum problem with differentiation, we found that the behavior changes completely at an angle of 30°. When the launch angle is smaller than 30°, there exists the optimal height depending on the angle, while the launch angle is larger than 30°, the board should be placed on the ground. This optimization problem will be a good exercise for undergraduate students in calculus-based physics courses.

REFERENCES

[1] Porter, W. S., *The range of a projectile*, Phys. Teach. **15**, 358 (1977).

[2] Lichtenberg, D. B. and Willis, J. G., *Maximizing the range of the shot put*, Am. J. Phys. **46**, 546 (1978).

[3] Bose, S. K., *Maximizing the range of the shot put without calculus*, Am. J. Phys. **51**, 458 (1983).

[4] Thomsen, J. S., *Maxima and minima without calculus*, Am. J. Phys. **52**, 881 (1984).

[5] Brown, R. A., *Maximizing the Range of a Projectile*, Phys. Teach. **30**, 344 (1992).