The Disk Paradox of Feynman

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Sepand Khosravi, **Houshyar Noshad**
Department of Physics and Energy Engineering, Amirkabir University of Technology, No. 424 Hafez Ave., Tehran, Iran.

E-mail: sepandkhosravi1@gmail.com

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Abstract

Special functions play a fundamental role in physics and engineering. In this work, we employ Legendre functions to analyze Feynman's disk paradox and demonstrate their utility in classical electrodynamics. The paradox concerns a non-conducting ring carrying a uniform charge distribution, placed in an external magnetic field. As the magnetic field is gradually reduced, the ring begins to rotate, seemingly violating angular momentum conservation. We show, however, that the electromagnetic field itself carries angular momentum. When the ring is stationary, this angular momentum is stored in the field; as the magnetic field decreases, it is transferred to the ring, ensuring that total angular momentum is conserved. This analysis not only resolves the paradox but also highlights the pedagogical and computational value of Legendre functions in electrodynamics

Keywords: Legendre Functions, Feynman's Disk Paradox, Electromagnetic angular momentum.

Resumen

Las funciones especiales desempeñan un papel fundamental en la física y la ingeniería. En este trabajo, empleamos funciones de Legendre para analizar la paradoja del disco de Feynman y demostrar su utilidad en la electrodinámica clásica. La paradoja se refiere a un anillo no conductor con una distribución de carga uniforme, situado en un campo magnético externo. A medida que el campo magnético se reduce gradualmente, el anillo comienza a girar, aparentemente violando la conservación del momento angular. Sin embargo, demostramos que el propio campo electromagnético posee momento angular. Cuando el anillo está estacionario, este momento angular se almacena en el campo; a medida que el campo magnético disminuye, se transfiere al anillo, garantizando la conservación del momento angular total. Este análisis no solo resuelve la paradoja, sino que también destaca el valor pedagógico y computacional de las funciones de Legendre en electrodinámica.

Palabras clave: Funciones de Legendre, Paradoja del Disco de Feynman, Momento angular electromagnético.

I. INTRODUCTION

Several methods have been proposed to resolve Feynman's disk paradox. Ma and Chiang [1] calculated the angular momentum stored in the electromagnetic field using direct integration. Torres del Castillo [2] argued that conservation can be established by a careful definition of angular momentum without explicit field calculations. Pantazis and Perivolaropoulos [3] considered a more realistic system with finite solenoid and charged cylinder. In contrast, the present work employs a Legendre polynomial expansion of the potential, which provides a systematic and symmetry-based approach.

The system studied in this work consists of a uniformly charged insulating ring of radius R placed in the x-y plane, which generates an electric field, together with a point magnetic dipole located in the same plane, providing the magnetic field. As the magnetic field is gradually reduced, the ring begins to rotate, which at first seems to challenge conservation of angular momentum. We show that the electromagnetic field itself carries angular momentum, which is transferred to the ring as the field decreases, ensuring total angular momentum is conserved [4].

II. ELECTRIC POTENTIAL OF THE CHARGED RING

We consider the charged ring of total charge Q located in the x-y plane. The electric potential at a point $\vec{r}$ is given by [5]:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\vec{r}-\vec{r}'|}\ , \qquad (1)$$

where ϵ_0 denotes the vacuum permittivity ($8.85\ \times 10^{-12}$ F/m).

We use the Legendre polynomial expansions [6]:

$$\frac{1}{|\vec{r}-\vec{r}'|}=\begin{cases}\sum_{l=0}^{\infty}P_l(\cos\gamma)\frac{r'^l}{r^{l+1}}\ , & r>r'\\ \sum_{l=0}^{\infty}P_l(\cos\gamma)\frac{r^l}{r'^{l+1}}\ , & r<r'\end{cases} \tag{2}$$

where P_l are the Legendre polynomials and γ is the angle between $\vec{r}$ and $\vec{r}'$.

The addition theorem for spherical harmonics [6] allows us to expand the Legendre polynomials:

$$P_l(\cos\gamma)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{l,m}(\theta,\varphi)Y_{l,m}^{*}(\theta',\varphi')\,, \tag{3}$$

where $Y_{l,m}$ are the spherical harmonics and the star denotes complex conjugation.

A. Electric Potential Outside the Charged Ring ($r > r'$)

We substitute Eq. (2) into the potential integral (Eq. (1)); then, by applying the addition theorem for spherical harmonics (Eq.(3)) and using $dq=\lambda dl=\lambda Rd\varphi'$ and $\theta'=\frac{\pi}{2}$ (since the ring lies in the x-y plane), we have:

$$d\Phi=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{l,m}(\theta,\varphi)Y_{l,m}^{*}(\theta',\varphi')\frac{r'^l}{r^{l+1}}\lambda Rd\varphi'\ . \tag{4}$$

Note that, during the integration, all terms vanish for all $m\neq 0$, so only $m=0$ contributes. Consequently, the sum over m is removed.

For points outside the ring ($r>R$), by integrating over the ring, we obtain:

$$\Phi=\frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{R^l}{r^{l+1}}P_l(\cos\theta)P_l(0).\quad r>R \tag{5}$$

It can be seen that, because the integration is carried out from 0 to 2π, $r'=R$.

B. Electric Potential Inside the Charged Ring ($r < r'$)

For points inside the charged ring ($r<R$) we use the expansion from (2), substitute it into the potential integral, apply the addition theorem for spherical harmonics, and use $dq=\lambda dl=\lambda Rd\varphi'$; then we have:

$$d\Phi=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{l,m}(\theta,\varphi)Y_{l,m}^{*}(\theta',\varphi')\frac{r^l}{r'^{l+1}}\lambda Rd\varphi', \tag{6}$$

by integrating over φ' from 0 to 2π as in the previous section, and using $r'=R$, we obtain:

$$\Phi=\frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)P_l(0)\ .\qquad r<R \tag{7}$$

III. ELECTRIC FIELD OF THE CHARGED RING

The electric field is obtained from the electric potential using [5]

$$\vec{E}=-\vec{\nabla}\Phi=\frac{\partial\Phi}{\partial r}\hat{r}-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}\,, \tag{8}$$

where the φ component vanishes due to azimuthal symmetry.

Using the results of Eqs. (5) and (7), we find the following components:

$$E_r=\begin{cases}\frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{(l+1)R^l}{r^{l+2}}P_l(\cos\theta)P_l(0), & r>R\\ -\frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{lr^{l-1}}{R^{l+1}}P_l(\cos\theta)P_l(0)\ , & r<R\end{cases} \tag{9}$$

$$E_\theta=\begin{cases}-\frac{Q}{4\pi\epsilon_0 r}\sum_{l=0}^{\infty}\frac{R^l}{r^{l+1}}P_l^1(\cos\theta)\,P_l(0)\,, & r>R\\ -\frac{Q}{4\pi\epsilon_0 r}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l^1(\cos\theta)\,P_l(0)\,, & r<R\end{cases} \tag{10}$$

where P_l and P_l^1 are the Legendre polynomials and associated Legendre functions, respectively.

Thus, the electric field of the charged ring is expressed analytically in terms of Legendre functions, valid for both the inside and outside regions.

IV. MAGNETIC FIELD OF THE DIPOLE

The system consists of a charged ring in the x-y plane, and a point magnetic dipole located at its center. The dipole is assumed to be sufficiently small compared to the ring, so that the magnetic dipole approximation is valid. Its magnetic moment is aligned along the z axis ($\vec{\mu}=\mu\hat{z}$).

In spherical coordinates, the unit vector along z is $\hat{z}=\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}$ [5], hence:

$$\vec{\mu}=\mu\big(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}\big). \tag{11}$$

We use the standard expression for a magnetic dipole [5]:

$$\vec{B}=\frac{\mu_0}{4\pi}\left[\frac{-\vec{\mu}+3(\vec{\mu}.\hat{r})}{r^3}\right], \tag{12}$$

where μ_0 denotes the vacuum permeability ($4\pi\times10^{-7}$ H/m).

By substituting Eq. (11) into (12), the resulting magnetic field is [4]:

$$\vec{B}=\frac{\mu_0\mu}{4\pi}\left[\frac{2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}}{r^3}\right]. \tag{13}$$

V.ANGULAR MOMENTUM OF THE ELECTRO-MAGNETIC FIELD

The electromagnetic momentum density is [5]:

$$\vec{P}_v = \epsilon_0 \vec{E} \times \vec{B} \ , \tag{14}$$

and the corresponding angular-momentum density is [5]:

$$\vec{L}_v = \vec{r} \times \vec{P}_v \, . \tag{15}$$

For the present geometry (the charged ring in the x-y plane and a point magnetic dipole at the ring center according to Eq. (11)), the electric and magnetic fields are azimuthally symmetric and have no φ components. Consequently, only the $\hat{\varphi}$ component of $\vec{P}_v$ is nonzero:

$$\vec{P}_v = \epsilon_0 (E_r B_\theta - E_\theta B_r) \hat{\varphi} \ , \tag{16}$$

hence,

$$\vec{L}_v = -r P_v \hat{\theta} = r \epsilon_0 (E_\theta B_r - E_r B_\theta) \hat{\theta} \, . \tag{17}$$

We write $\hat{\theta}$ in Cartesian components and integrate over φ from 0 to 2π, it shows that the components of the angular momentum is zero in x and y direction, therefore the total field angular momentum $L_z = \epsilon_0 \int \left(\vec{r} \times \left(\vec{E} \times \vec{B} \right) \right) . \hat{z}$ points along z, which is represented as follows:

$$L_z = 2\pi \epsilon_0 \int_0^\infty r^3 dr \int_0^\pi (E_r B_\theta - E_\theta B_r) sin^2\theta \ d\theta . \tag{18}$$

We split the radial radial integral into the inner region ($0 < r < R$) and the outer region ($r > R$):

$$L_z = L_z^{(<)} + L_z^{(>)} \ , \tag{19}$$

where,

$$L_z^{(<)} = 2\pi \epsilon_0 \int_0^R r^3 dr \int_0^\pi (E_r B_\theta - E_\theta B_r) sin^2\theta \ d\theta \ , \tag{20}$$

$$L_z^{(>)} = 2\pi \epsilon_0 \int_R^\infty r^3 dr \int_0^\pi (E_r B_\theta - E_\theta B_r) sin^2\theta \ d\theta . \tag{21}$$

A. The inner contribution of the angular momentum

The inner contribution of the angular momentum $L_z^{(<)}$ is evaluated by defining:

$$a = \int_0^R r^3 dr \int_0^\pi E_r B_\theta \sin^2\theta \ d\theta \, , \tag{22}$$

$$b = \int_0^R r^3 dr \int_0^\pi E_\theta B_r \sin^2\theta \ d\theta \ , \tag{23}$$

Hence $L_z^{(<)} = 2\pi \epsilon_0 (a - b)$. We substitute electric and magnetic field components from Eqs. (9), (10), and (13) into Eqs. (22) and (23) and we have

$$a = -\frac{Q \mu_0 \mu}{(4\pi)^2 \epsilon_0} \sum_{l=0}^{\infty} \frac{l P_l(0)}{R^{l+1}} \int_0^R dr \, r^{l-1} \int_0^\pi P_l(cos\theta) \sin^3\theta \, d\theta , \tag{24}$$

$$b = -\frac{2Q \mu_0 \mu}{(4\pi)^2 \epsilon_0} \sum_{l=0}^{\infty} \frac{P_l(0)}{R^{l+1}} \int_0^R dr \, r^{l-1} \int_0^\pi P_l^1(\cos\theta) \cos\theta \sin^2\theta \ d\theta \ , \tag{25}$$

then we evaluate the angular integrals using the orthogonality relations for Legendre polynomials and associated Legendre functions (Eqs. (26) and (27)) [6]:

$$\int_0^\pi P_n(\cos\theta) P_m(\cos\theta) \ \sin\theta \, d\theta = \frac{2}{2n+1} \delta_{n,m} , \tag{26}$$

$$\int_0^\pi P_l^m(\cos\theta) P_{l'}^m(\cos\theta) \sin\theta \ d\theta \ = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l,l'} \ . \tag{27}$$

It should be emphasized that in Eq. (24), only $l = 0$ and $l = 2$ terms survive the angular integral and in Eq.(25) only $l = 2$ terms survives.

After that, we perform the radial integrals and we have:

$$a = -\frac{2\mu_0 Q \mu}{15(4\pi)^2 \epsilon_0 R} \, , \tag{28}$$

$$b = -\frac{6\mu_0 Q \mu}{15(4\pi)^2 \epsilon_0 R} \ , \tag{29}$$

note that in Eq.(24), the radial integral is zero for $l = 0$ and therefore we only used $l = 2$ for the result.
By substituting Eqs.(28) and (29) into Eq.(20) we have:

$$L_z^{(<)} = \frac{2}{15} \frac{\mu_0 Q \mu}{4\pi R} \ . \tag{30}$$

B. The outer contribution of the angular momentum

Similarly, the outer contribution $L_z^{(>)}$ is obtained by defining:

$$c = \int_R^\infty r^3 dr \int_0^\pi E_r B_\theta \sin^2\theta \, d\theta , \tag{31}$$

$$d = \int_R^\infty r^3 dr \int_0^\pi E_\theta B_r \sin^2\theta d\theta \ , \tag{32}$$

therefore $L_z^{(>)} = 2\pi\epsilon_0(c-d)$. By substituting electric and magnetic field components from Eqs. (9), (10), and (13) into Eqs.(31) and (32) we have:

$$c = \frac{Q\mu_0\mu}{(4\pi)^2\epsilon_0}\sum_{l=0}^{\infty}(l+1)R^l P_l(0)\int_R^{\infty}\frac{dr}{r^{l+2}}\int_0^{\pi}P_l(cos\theta)\sin^3\theta\, d\theta\ , \qquad (33)$$

$$d = -\frac{2Q\mu_0\mu}{(4\pi)^2\epsilon_0}\sum_{l=0}^{\infty}P_l(0)R^l\int_R^{\infty}\frac{dr}{r^{l+2}}\int_0^{\pi}P_l^1(\cos\theta)\cos\theta\ \sin^2\theta\, d\theta\ , \quad (34)$$

by evaluating the integrals in Eqs. (33) and (34), similar to the previous subsection, we obtain:

$$c = \frac{22Q\mu_0\mu}{15(4\pi)^2\epsilon_0 R}\,, \qquad (35)$$

$$d = -\frac{4Q\mu_0\mu}{15(4\pi)^2\epsilon_0 R}\,, \qquad (36)$$

by substituting Eqs. (35) and (36) into (21) we have:

$$L_z^{(>)} = \frac{13}{15}\frac{\mu_0 Q\mu}{4\pi R}\,. \qquad (37)$$

C. The total angular momentum

By adding $L_z^{(<)}$ from Eq. (30) and $L_z^{(>)}$ from Eq. (37), we can have the total angular momentum of the electromagnetic field:

$$L_z = \frac{2}{15}\frac{\mu_0 Q\mu}{4\pi R} + \frac{13}{15}\frac{\mu_0 Q\mu}{4\pi R} = \frac{\mu_0 Q\mu}{4\pi R}\ \ . \qquad (38)$$

VI. MECHANICAL ANGULAR MOMENTUM OF THE CHARGED RING

In this section, we reduce the magnetic dipole moment and, consequently, the magnetic field, and evaluate the mechanical angular momentum transferred to the charged ring as the magnetic field vanishes. To this end, we first calculate the magnetic flux passing through the ring. By using Faraday's law of induction, we obtain the induced electric field due to the time-varying magnetic field. This induced field produces a torque on the charged ring, from which the transferred mechanical angular momentum is determined.

A. Magnetic Flux Through the Ring

The magnetic flux through the ring is expressed as [5]:

$$\Phi_B = \int \vec{B}\,.\,d\vec{A} = \int B_z\, dA = 2\pi\int_0^R B_z\, r dr\ \ . \qquad (39)$$

Here, dA is the surface element of the ring. Since the ring lies in the x-y plane, the normal vector is along the z axis. From the spherical representation of the dipole magnetic field (Eq. (13)), the z component is given by [4]:

$$B_z = \cos\theta B_r - \sin\theta B_\theta = \frac{\mu_0\mu}{4\pi r^3}(3\cos^2\theta - 1)\ , \qquad (40)$$

using the Legendre polynomial relation $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ [6], Eq. (40) can be written as:

$$B_z = \frac{\mu_0\mu}{2\pi}\frac{P_2(\cos\theta)}{r^3}\,. \qquad (41)$$

Since the ring lies in the plane $\theta = \frac{\pi}{2}$, we have $P_2(0) = -\frac{1}{2}$. Substituting into Eq. (39) yields:

$$\Phi_B = -\frac{\mu_0\mu}{2}\int_0^R\frac{dr}{r^2}\ . \qquad (42)$$

The integral in Eq. (43) diverges as $r \to 0$, which is the consequence of approximating the source as a point dipole located at the origin. To overcome this difficulty, we note that the total magnetic flux of a dipole through an infinite plane is zero. Thus, the flux through the inner region can be obtained as the negative of the flux through the exterior region [4]:

$$\int_R^{\infty} B_z\,(2\pi r)dr = -\int_0^R B_z\,(2\pi r)dr\,, \qquad (43)$$

evaluating the left-hand side gives by substituting B_z from Eq. (41) by using $\theta = \frac{\pi}{2}$:

$$\int_R^{\infty} B_z\,(2\pi r)dr = -\frac{\mu_0\mu}{2R}\,. \qquad (44)$$

Therefore, the magnetic flux through the ring is:

$$\Phi_B = \frac{\mu_0\mu}{2R}\,. \qquad (45)$$

B. Mechanical Angular Momentum of the Ring

The reduction of the magnetic dipole moment induces an electric field according to Faraday's law Eq.(46) [5],

$$\oint \vec{E}.d\vec{l} = -\frac{d\Phi_B}{dt}\,. \qquad (46)$$

Since the induced field is azimuthal:

$$E_\varphi(2\pi r) = -\frac{\mu_0}{2R}\frac{d\mu}{dt} \to E_\varphi = -\frac{\mu_0}{4\pi R^2}\frac{d\mu}{dt}\,. \qquad (47)$$

This induced field exerts a force on the charged ring [5]:

$$\vec{F} = Q\vec{E} = -\frac{\mu_0 Q}{4\pi R^2}\frac{d\mu}{dt}\hat{\varphi}\,, \tag{48}$$

which produces a torque as follows [5]:

$$\vec{\tau} = \vec{r} \times \vec{F} \xrightarrow{\vec{r}=R\hat{r}} \vec{\tau} = \frac{\mu_0 Q}{4\pi R}\frac{d\mu}{dt}\hat{\theta}\quad, \tag{49}$$

note that $\hat{\theta}$ is the unit vector in the spherical coordinates [5]:

$$\hat{\theta} = \cos\theta\cos\varphi\hat{x} + \cos\theta\sin\varphi\hat{y} - \sin\theta\hat{z}\ , \tag{50}$$

since the ring is located in the x-y plane $\theta = \frac{\pi}{2}$, therefore from Eqs. (49) and (50) we have:

$$\vec{\tau} = -\frac{\mu_0 Q}{4\pi R}\frac{d\mu}{dt}\hat{z}\ \ . \tag{51}$$

The mechanical angular momentum transferred to the ring is obtained from [4]:

$$\vec{\tau} = \frac{d\vec{L}_{mech}}{dt} \rightarrow \vec{L}_{mech} = \int_{ti}^{t_f} \vec{\tau}\,dt = -\frac{\mu_0 Q}{4\pi R}\int_{\mu}^{0} d\mu\,\hat{z}\quad, \tag{52}$$

which yields:

$$L_{mech,z} = \frac{\mu_0 Q\mu}{4\pi R}\,. \tag{53}$$

It should be noted that the reduction of the magnetic field must occur sufficiently slowly so that the system remains within the quasi-magnetostatic regime and radiation effects can be neglected.

VII. CONCLUSION

In this work, we calculated the electromagnetic field generated by a charged, non-conducting ring and a point magnetic dipole, and from this we obtained the angular momentum of the electromagnetic field (Eq. (38)). By gradually reducing the magnetic dipole moment, we demonstrated that the charged ring starts to rotate. The mechanical angular momentum acquired by the ring after the dipole moment vanishes (Eq. (53)) is exactly equal to the angular momentum of the electromagnetic field given in Eq. (38), namely $\frac{\mu_0 Q\mu}{4\pi R}$.

This indicates that there is no inconsistency: the apparent violation of angular momentum conservation is resolved once the angular momentum stored in the electromagnetic field is properly taken into account. Furthermore, when $\mu = 0$, the magnetic field vanishes according to Eq. (13), and therefore the electromagnetic angular momentum also becomes zero. This confirms that the entire angular momentum is transferred to the ring.

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