One or two magnets falling in a conductive pipe: On-axis and off-axis fall and the role of the pipe wall thickness

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Abstract
The retarded vertical motion of a short cylindrical magnet falling inside a *non-magnetic* conductive pipe is an appealing phenomenon that students, teachers and the public at large enjoy as in a magician performance. This content rich phenomenon has been recently addressed a number of times in physics teaching journals. The retarded fall of the magnet is seen to consist of an initial transient accelerated regime followed by motion at a terminal speed. After the phenomenon is demonstrated many interesting questions and useful suggestions quickly arise during the ensuing discussion. What would happen if the magnet falls off the pipe axis? Why don't you try with two magnets at the time? In this work we present theoretical models and experimental results for this retarded fall, considering first the case of conductive pipes of different materials and different wall thickness, and then the case of two magnets falling together. We here predict with good accuracy the experimental results later obtained by us in the laboratory. The effect of varying the pipe wall thickness on the magnetic drag is studied for pipes of two different materials, copper and aluminium. The case of magnetic braking in aluminium pipes is interesting since copper is diamagnetic while aluminium is paramagnetic We have performed experiments with the two magnets falling together either with *parallel* or *opposite magnetic moments*, with variable separation in between, and also experiments with a single magnet falling at different distances from the pipe axis. We develop successful analytical models for these cases of magnetic braking. The experimental setups are inexpensive and can be readily assembled in a teaching laboratory.

Keywords: Foucault currents, Magnetic braking, Faraday Law of Induction.

Resumen
La caída vertical retardada de un pequeño imán cilíndrico a lo largo de un tubo conductor y no magnético es un llamativo fenómeno físico que los estudiantes, profesores y público en general disfrutan como si se tratase de un acto de magia. Este fenómeno, de mucho contenido físico, ha sido presentado varias veces recientemente en revistas de enseñanza de la física. La caída retardada del imán consiste de un movimiento acelerado transitorio inicial que es seguido de caída uniforme con rapidez terminal. Luego de que este fenómeno es mostrado a los estudiantes surgen de inmediato un número de preguntas y de sugerencias interesantes. ¿Qué pasará si el imán cae alejado del eje vertical de simetría del tubo conductor? ¿Podríamos dejar caer dos imanes juntos a ver qué pasa? En este trabajo se presentan los modelos teóricos y los resultados experimentales de este fenómeno de movimiento retardado en un tubo conductor, primero cuando se consideran tubos conductores de diferente espesor y luego cuando se dejan car dos imanes juntos a lo largo del tubo conductor. Predecimos con buena exactitud los resultados experimentales que luego obtuvimos en el laboratorio. El efecto de la variación del espesor de las paredes del tubo es estudiado con tubos de diferente material, cobre y aluminio. El frenado magnético en aluminio es interesante pues este metal es paramagnético, mientras que el cobre es diamagnético. Hemos realizado experimentos con dos imanes que caen con sus momentos dipolares paralelos y opuestos, y además variando la separación entre ellos, y experimentos con un solo imán que cae a diferentes distancias del eje del tubo. Aquí desarrollamos modelos analíticos exitosos para todos estos casos de frenado magnético. Los montajes experimentales son económicos y pueden ser armados fácilmente en cualquier laboratorio de física.

Palabras clave: Frenado magnético, Corrientes de Foucault, Ley de Faraday para la Inducción Electro-magnética.

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I. INTRODUCTION

The magnetic braking of a falling magnet in a conducting pipe is an appealing experiment that surprise students, and the public in science museums. It is also an intriguing phenomenon for anyone to explain. The subject has already been recently addressed in scientific journals a number of times [1 (and references therein), 2]. Magnetic braking has found in addition a number of important applications in present technology. The vertical motion of a button-shaped magnet inside a conducting tube has recently begun to be studied thoroughly in the laboratory [1, 2]. A small magnet falling inside a vertical conducting pipe is just another example of a body that falls under the action of a retarding force; the case of sky-divers, and of the remarkable "flying" seeds of some trees [3], being other interesting cases of motion under the action of retarding forces for physics students. In all these instances the body begins falling in transient accelerated motion, a regime that quickly yields to uniform motion under the action of the weight and the retarding force. The body speed in this uniform motion regime is the so-called *terminal speed*. In Section II we again consider magnetic braking, and derive the pertinent equations of the phenomenon in a new perspective, and additionally examine the role of the thickness of the pipe wall, both experimentally and theoretically. The results of experiments with pipes of different wall thicknesses, and two different materials, are reported (Section III) to check the validity of the models here developed. We consider (Section IV) the case when two magnets, separated a given distance, fall along the pipe, and in Section V when the magnet fall off the pipe axis.

II. MAGNETIC DRAGGING FORCE AND THE PIPE WALL THICKNESS

Fig. 1 is a vertical cross-section of a button-shaped magnet as it falls with velocity **v** inside a copper pipe of radius a. Foucault circular currents of intensity di are induced in an infinitesimal ring of height dz. We first derive an expression for the magnetic braking dF force exerted on the falling magnet by the infinitesimal ring.

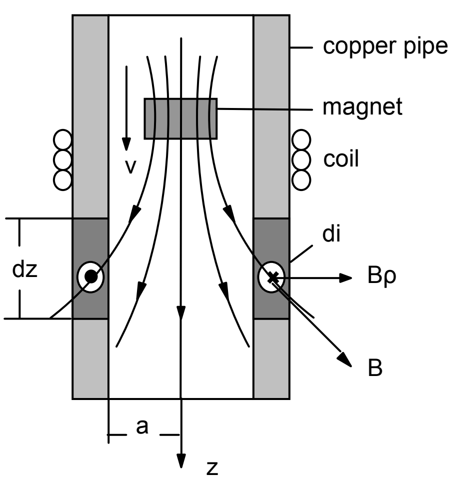

FIGURE 1. Vertical cross-section of a magnet falling inside a conducting pipe. A circular current di is induced in a pipe ring of height dz. A small external coil is used to measure the induced *e.m.f.* $\mathbf{B}_\rho$ is the radial component of the magnet field **B**.

The *motional e.m.f.* ε_i induced in the pipe is given by the well-known relation

$$\boldsymbol{\varepsilon_i} = \int \left(\boldsymbol{v} \times \boldsymbol{B_\rho}\right) \cdot \boldsymbol{dl} = \boldsymbol{2\pi a v B_\rho(a, z)}, \qquad (1)$$

where $B_\rho(a, z)$ is the radial component of the magnetic field at the pipe wall and distance z [1, 4]. If μ is the magnetic dipole of the magnet this radial component is given by

$$B_\rho(a, z) = \frac{3\mu z\rho}{[a^2+z^2]^{5/2}}. \qquad (2)$$

Let σ be the conductivity of the material, e the pipe thickness. The induced electrical current di present in the infinitesimal ring of length $2\pi a$ of Fig. 1 is therefore given by (see Eq. 1)

$$di = \frac{\sigma(e\, dz)}{2\pi a}\varepsilon_i = (2\pi a)\, v\, B_\rho(a, z)\, \frac{\sigma e\, dz}{2\pi a},$$

$$di = v\, \sigma\, e\, B_\rho(a, z) dz. \qquad (3)$$

The magnetic force dF on the infinitesimal ring (given by the well-known relation $l\, B\, di$) is written as

$$dF = (2\pi a)\sigma e v B_\rho^2(a, z)\, dz, \qquad (4)$$

or after using the radial component of the field in Eq. (2) and defining a new variable u such that $z = au$

$$dF = 2\pi a^2 \sigma e v \left(\frac{3\mu}{a^3}\right)^2 \frac{u^2}{(1+u^2)^5} du. \qquad (5)$$

Integrating along the pipe we finally get the sought magnetic retarding force on the magnet

$$F = 2\pi a^2 \sigma e v \left(\frac{3\mu}{a^3}\right)^2 \int_0^\infty \frac{u^2}{(1+u^2)^5} du, \qquad (6)$$

$$F = 2\pi a^2 \sigma e v \left(\frac{3B_0}{a^3}\right)^2 I, \qquad (7)$$

where

$$I = \int_0^\infty \frac{u^2}{(1+u^2)^5} du = \frac{5\pi}{256} \quad \text{and} \quad B_0 = {}^{2\mu}\!\Big/_{z_0^3}, \qquad (8)$$

is just the magnetic field at the axial point of coordinate z_0 from the magnet. Therefore the dragging force on the falling magnet is just proportional to the magnet speed v

$$F = (36\pi I e \sigma \mu^2) v = kv, \qquad (9)$$

where $k = 36\pi I e \sigma \mu^2 / a^4$ is just the *dragging* or *viscous constant*. We therefore expect the terminal speed of the magnet to be inversely proportional to the wall thickness e.

It is now easy to find an expression for the terminal speed v_T of the magnet. Newton´s Second Law gives:

$$m\, {dv}/{dt} = mg - kv, \tag{10}$$

whose solution is readily found

$$v(t) = v_T\left[1 - \exp\left(-{t}/{\tau}\right)\right]; \quad v_T = mg/k\,, \tag{11}$$

where v_T is the terminal of the magnet.

III. EXPERIMENTS WITH PIPES OF DIFFERENT WALL THICKNESS

Nine 15cm long pipes of different diameters and wall thicknesses – 5 of them made of copper and 4 of aluminium – were machined down from commercially available pipes (neither made of pure metals). Their inner diameter is 13.8mm while their external diameters are 15.8, 18.0, 20.0, 22.0 and 25.4mm. A cylindrical Nd/FeNi magnet, 12.7mm in diameter and 3.6mm long, mass 3.0g, was used in all the experiments.

In Fig. 2 we have plotted the vertical position of the magnet as it falls inside the 5 copper pipes of increasing wall thickness *e*. The continuous lines are given by our theoretical model. It may be seen that the agreement between our theory and the experiments is very good. Indeed the terminal speed of the magnet is inversely proportional to the thickness.

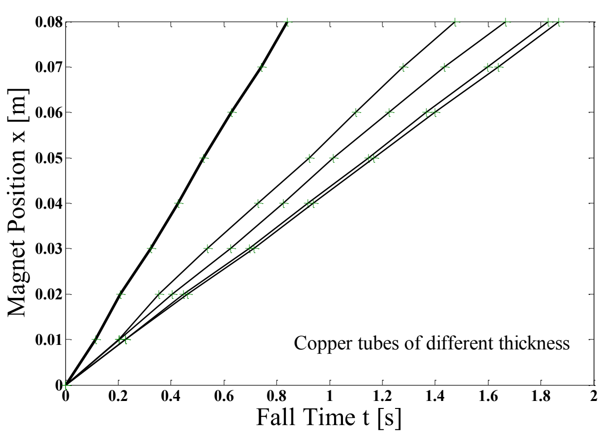

FIGURE 2. Vertical position of the falling magnet as a function of time. The straight lines, from left to right correspond to *copper* pipes of increasing wall thickness e = 2.00, 2.10, 3.10, 4.10 and 5.80mm, respectively. The slope of each line is the terminal speed of the magnet; it decreases as the wall thickness increase.

In Fig. 3 we have plotted the vertical position of the magnet as it falls inside 4 aluminium pipes of increasing wall thickness. The continuous lines are given by our theoretical model. It may be seen that the agreement between theory and experiments is again very good. Again the terminal speed of the magnet is inversely proportional to the wall thickness. The terminal speeds are definitely higher in aluminium than in copper.

IV. TWO MAGNETS FALLING ALONG THE PIPE

Let us now consider now the case of two short identical cylindrical magnets falling together inside the conductive pipe, along its axis. The magnets are separated (Fig. 4) by a vertical distance *s*– fixed with ceramic disks– and with their magnetic moments *in parallel*, *i.e. South Pole* of the upper magnet facing *North Pole* of the lower one.

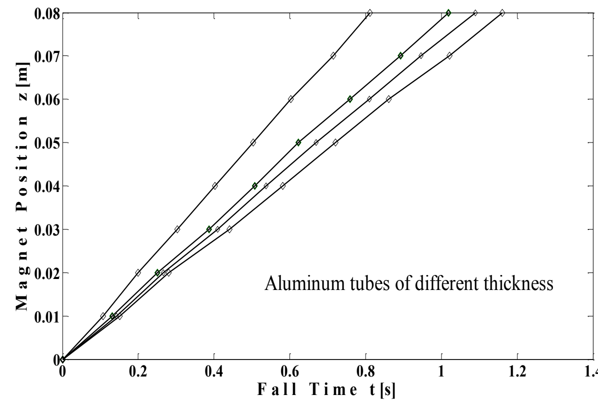

FIGURE 3. Vertical position of the falling magnet as a function of time. The straight lines, from left to right correspond to aluminium pipes of increasing wall thickness e = 2.10, 3.10, 4.10 and 5.80mm, respectively. The slope of each line is the terminal speed of the magnet; it decreases as the wall thickness increase.

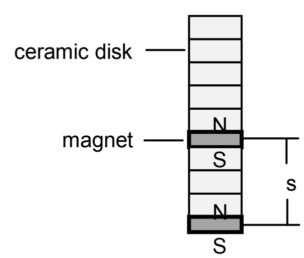

FIGURE 4. Two magnets separated by a variable number of ceramic disks placed in between (the total number of these disks is kept the same so that the total weight of the assembly is kept constant).

As in Section I the dragging force on the object is given by

$$F = 2\pi a v\, \sigma e \int_{-\infty}^{\infty} B_\rho^2(a,z)\, dz. \tag{12}$$

The radial component of the field of a single magnet is now written as

$$B_\rho(a,z) = \frac{3\tilde{\mu}\,\alpha z\, a}{[a^2+(\alpha z)^2]^{5/2}}\,, \tag{13}$$

where $\tilde{\mu} \equiv (\mu_0/4\pi)\mu$, and μ represents the actual magnetic dipole of the magnet in S.I. unit. α is simply a convenient parameter and the quantity $\tilde{\mu}$ [Tm^3] is introduced just to simplify the algebraic expressions that follow. Both $\tilde{\mu}$ and α are found in a preliminary and very easy experiment [see the Appendix of Ref. [4]. Thus for the two falling magnets - separated the distance s in between and parallel dipoles- we write

$$B_\rho(a,z) = \frac{3\tilde{\mu}a\alpha(z-s/2)}{[a^2+\alpha^2(z-s/2)^2]^{5/2}} + \frac{3\tilde{\mu}a\alpha(z+s/2)}{[a^2+\alpha^2(z+s/2)^2]^{5/2}}, \quad (14)$$

and with the change of variables $u=\alpha\, z/a$

$$B_\rho(a,u) = \frac{3\tilde{\mu}}{a^3}\left[\frac{u-(\frac{\alpha s}{2a})}{\left[1+\left(u-(\frac{\alpha s}{2a})\right)^2\right]^{5/2}} + \cdots\right.$$

$$\left.\frac{u+(\frac{\alpha s}{2a})}{\left[1+\left(u+(\frac{\alpha s}{2a})\right)^2\right]^{5/2}}\right]. \quad (15)$$

As in Section II (See Eqs. (5) to (8)) we need to integrate the bracketed expression in the *r.h.s.* of Eq. (15):

$$I_{par}({}^{S}\!/_{a}) = \int_0^\infty \left[\frac{u-(\frac{\alpha\, s}{2a})}{\left[1+\left(u-(\frac{\alpha\, s}{2a})\right)^2\right]^{5/2}} + \right.$$

$$\left.\frac{u+(\frac{\alpha\, s}{2a})}{\left[1+\left(u+(\frac{\alpha\, s}{2a})\right)^2\right]^{5/2}}\right]^2 du. \quad (16)$$

With this we get an expression for the dragging force on our assembly of two magnets falling together (case of *parallel dipoles*)

$$F \; = \frac{36\pi\sigma e\, \tilde{\mu}^2}{\alpha a^4} I\left(\frac{s}{a}\right) v = k\, v\,, \quad (17)$$

as well as the predicted terminal speed $v_T = 2mg/k$ of the assembly.

V. EXPERIMENTS WITH TWO MAGNETS FALLING TOGETHER

Fig. 5 shows our experimental results for the termmeinal speed when two button-shaped Nd: FeB magnets (mass 3g, 3.15mm thick, 12.7mm diameter) assembled as in Fig. 4 were allowed to fall inside a copper pipe (10.4mm internal radius, 1.3 wall thickness). The mass of the assembly was 14g (magnets assembled with *parallel magnetic dipoles*). The continuous curve represents the terminal speed predicted by our model (Eqs. (17, 18)). The agreement between the experimental data and theory is good.

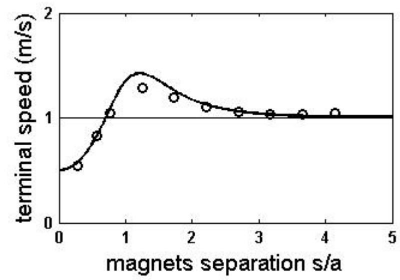

FIGURE 5. Terminal speed of two magnets falling together inside a copper pipe of radius *a*, with their magnetic dipoles in parallel. The magnets separation is *s*.

Fig. 6 shows experimental results for the terminal speed when the magnets fell with *magnetic dipoles anti-parallel*. The continuous curve represents the terminal speed predicted by our model [4]. The variation of the terminal speed with the dimensionless variable *s/a* is the result of the partial cancellation of the magnets magnetic fields. In both Fig. 5 and Fig. 6 as *s* gets larger the terminal speed tends to the terminal speed of a single falling magnet.

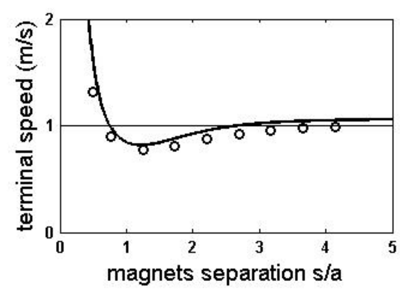

FIGURE 6. Terminal speed of two magnets falling together inside a copper pipe of radius *a,* with their magnetic dipoles anti-parallel. The magnets separation is *s*.

VI. MAGNET FALLING OFF-AXIS

Fig. 7 depicts the cylindrical magnet falling, with velocity **v**, along a vertical guiding glass tube (z'-axis) at distance b from the symmetry z-axis of the conductive pipe. The falling magnet induces an *e.m.f.* in a ring-shaped differential element of height dz of the pipe below it. As in Section II the magnet field is decomposed into a component parallel to the pipe axis, and a radial component $\boldsymbol{B}_\rho$ from the centre *O'*, of the glass tube. $P(a,\theta)$ is a point of the infinitesimal ring as seen from the centre *O* (Fig. 8) of the ring, and ρ is the polar radius of *P* as seen from the point *O'* on the z'-axis. For the off-axis fall the radial component $\boldsymbol{B}_\rho$ of the field forms a variable angle $\psi(\rho,\, b)$ with the radius vector *OP* from the centre of the infinitesimal ring.

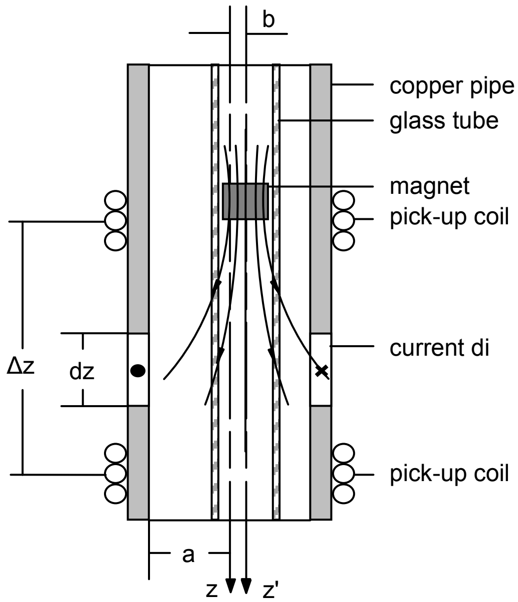

FIGURE 7. A single magnet falling off-axis along a conductive pipe (a glass tube is used to guide the magnet).

The cross product $\mathbf{v}\times\boldsymbol{B}_\rho$ is not tangential to the pipe wall ($\mathbf{v}$ is vertical the magnet velocity). Note the variable angle $\psi(\rho, b)$.

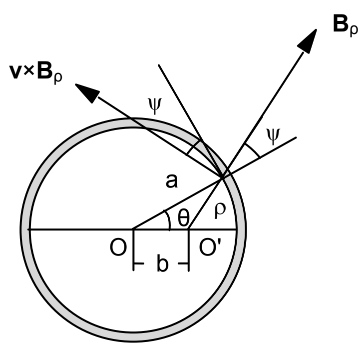

FIGURE 8. The magnitude B_ρ of the radial component is now a function of the distance ρ, and of the vertical coordinate z of the infinitesimal ring.

The induced *e.m.f.* on the now eccentric infinitesimal conducting ring is now given by (cf. Section II)

$$\varepsilon_i = \int_0^{2\pi} v\, B_\rho\,(\rho, z)\ \cos\,\psi\,(\rho, b)\; a\, d\theta, \tag{18}$$

and from the geometry of Fig. 8 we may write,

$$\cos\psi = \frac{a - b\cos\vartheta}{\rho},$$

and

$$\rho^2 = a^2 + b^2 - 2ab\cos\theta.$$

The radial component of the magnetic field is now given by an equation equivalent to Eqs. (2 and 13) [4]:

$$B_\rho = \frac{3\tilde{\mu}\rho(\alpha z)}{[\rho^2 + (\alpha z)^2]^{5/2}}, \tag{19}$$

where again $\tilde{\mu}$ (in units of Tm^3) is the constant defined in Section III above, and α is the adjusting parameter of our approximation. With the expression given in Eq. (19) for the vector component $\boldsymbol{B}_\rho$ the induced *e.m.f.* becomes,

$$\varepsilon_i = \int_0^{2\pi} v \frac{3\tilde{\mu}\rho(\alpha z)}{[a^2 + b^2 - 2ab\cos\theta + (\alpha z)^2]^{5/2}} \times \ldots$$

$$\frac{(a - b\cos\theta)}{\rho} a d\theta. \tag{20}$$

an integral on the polar angle θ. Introducing a new dimensionless variable u such that $z = au/\alpha$ the last equation becomes,

$$\varepsilon_i = \frac{6\pi\tilde{\mu}v}{a^2} G(u, b), \tag{21}$$

$$G(u, b) = \int_0^{2\pi} \frac{u\,[1 - (b/a)\cos\theta]}{2\pi\left[1 + (\frac{b}{a})^2 - 2(\frac{b}{a})\cos\theta + (u)^2\right]^{5/2}} d\theta, \tag{22}$$

to be numerically evaluated. The dragging force dF applied to the magnet by the infinitesimal ring is now

$$dF_z = \frac{\sigma e v du}{2\pi\alpha}\left(\frac{6\tilde{\mu}\pi}{a^2}\right)^2 [G(u, b)]^2, \tag{23}$$

and the force on the falling magnet is now given by

$$F_z = \frac{36\,\pi\sigma e v \tilde{\mu}^2}{\alpha\, a^4}\int_0^\infty [G(u, b)]^2\, du \equiv \frac{36\,\pi\sigma e v\tilde{\mu}^2}{\alpha\, a^4}\tilde{f}(b/a)$$

Fig. 9 shows our experimental results for the magnet falling off-axis (distance b from the pipe axis). It may be seen that the terminal speed decreases as the magnet falls closer to the pipe wall (greater dragging force). The continuous line is the prediction of our analytical model. The agreement is very good.

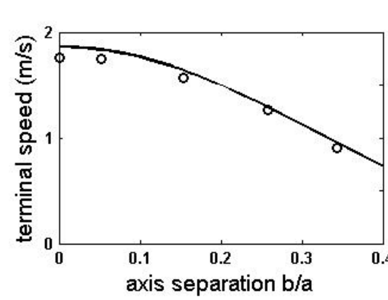

FIGURE 9. Terminal speed of a magnet falling off-axis inside a copper pipe of radius a. The separation from the pipe axis to the magnet path is b.

VII. CONCLUSIONS

We have developed successful theoretical models for the magnetic dragging force on a magnet that falls inside a conductive pipe. These models predict the laboratory measured terminal speed of the magnets when the thickness of the pipe wall is varied, when two magnets fall together and when the magnet falls off-axis. The experiments described above confirmed the theretical models, and can be readily setup in any undergraduate physics laboratory. They are highly recommendable for project work for both physics and engineering students.

REFERENCES

[1] Donoso, G., Ladera, C. L. and Martin, P., *Magnet fall inside a conductive pipe: motion and the role of the pipe wall thickness*, Eur. J. Phys. **30**, 855-869 (2009).

[2] Ireson, G. and Twiddle, J., *Magnetic braking revisited: Activities in the undergraduate laboratory*, Eur. J. Phys. **29**, 557-565 (2009).

[3] Ladera, C. L. and Pineda, P., *The physics of the spectacular flight of the Triplaris samaras*, Lat. Am. J. Phys. Educ. **3**, 952-968 (2009).

[4] Donoso, G., Ladera, C. L. and Martin, P., *Damped fall of magnets inside a conducting pipe: A magnet falling close to the pipe wall and two magnets falling together*, Am. J. Phys. **79**, 193-200 (2011).