The beauty and power of symmetry in Physics

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Abstract
Symmetry is a crucial concept – not only in Math and Physics, but also in Chemistry, Biology, art, architecture and music. It adds an element of beauty to Physics. Einstein spoke of his "rapturous amazement at the harmony of natural law". Symmetry is also very powerful in simplifying calculations. Using two home-made examples with varying charge density, one linear and the other circular, we show how we can communicate to the students the beauty and power of symmetry – with calculus, and even without it!

Keywords: Symmetry, Beauty, General physics, Teaching methods and strategies, Learning theory and science teaching.

Resumen
La simetría es un concepto crucial – no sólo en Matemáticas y Física, sino también en Química, Biología, Arte, Arquitectura y Música. Se añade un elemento de belleza a la Física. Einstein habló de su "asombro extasiado ante la armonía de la ley natural". La simetría es también muy potente en la simplificación de los cálculos. El uso de dos ejemplos de fabricación casera con la densidad de carga variable, una lineal y la otra circular, nos muestran cómo se puede comunicar a los estudiantes de la belleza y el poder de la simetría – con el cálculo, y aún sin ella!

Palabras clave: Simetría, Belleza, Física general, Enseñanza de métodos y estrategias, Teorías de aprendizaje y enseñanza de la ciencia.

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I. INTRODUCTION

Symmetry is the invariance of an object or system to a set of changes/transformations. In layperson's language, an object possesses symmetry if it looks identical before and after a change. For example, when we rotate a rectangular piece of cardboard by 180^o with respect to the axis through its center and perpendicular to the plane of the cardboard, its appearance after the rotation is the same as before. We say that the rectangular cardboard has a 2-fold ($360^o/180^o$) rotational symmetry. Similarly, a square cardboard is invariant under a 90^o rotation and hence has a 4-fold ($360^o/90^o$) symmetry.

II. THE IMPORTANCE & BEAUTY OF SYMMETRY

The importance of symmetry has been described very well by Hill and Lederman [1] in their article, "Teaching Symmetry in the Introductory Physics Curriculum". They show how the symmetries dictate the basic laws of Physics. Further, these authors have established the web site www.emmynoether.com on the same topic.

According to the poet Keats [2],
"*Beauty is truth, truth beauty" – that is all*
Ye know on earth, and all ye need to know".

It is nice to compare this with the words of Heisenberg [3] to Einstein, "If nature leads us to mathematical forms of great simplicity and beauty … we cannot help thinking that they are 'true'; that they reveal a genuine feature of nature". While searching for beauty in truth, symmetry comes in very handy. For, as Paul Davies [4] puts it, "Central to the physicist's notion of beauty are harmony, simplicity and symmetry". Many would agree with the words of Hill and Lederman [1]: "Symmetry is one of the most beautiful concepts, and its expression in nature is perhaps the most stunning aspect of our physical world".

III. THE POWER OF SYMMETRY

In communicating the importance and beauty of symmetry to the students doing Introductory Physics, we have found it very helpful to stress the power i.e. usefulness of symmetry to them. The students at that level are more interested in getting a good grade. Showing to them how symmetry can simplify calculations in the problems in their tutorials, tests and exams is very important. This would mean that we incorporate into our lectures, tutorials etc. specially tailored problems which involve simplifications due to symmetry.

IV. PROBLEMS USING SYMMETRY FOR SIMPLIFICATION

A. A Circular Wire with Angular Variation of λ

Consider a circular wire of radius R and net charge +Q, with $\lambda=\lambda_o\,\theta$. Our aim is to calculate the direction and magnitude of the net electric field **E** at the center.

Using elementary calculus, it is easy to show that $\lambda_o=Q/2\pi^2R$. Considering the cancellation of **E** at the centre due to pairs of oppositely placed charge elements, as seen in Fig. 1, we see that

$$dq' \text{ at } (\theta+\pi) - dq \text{ at } \theta=\lambda_o(\theta+\pi)(R\,d\theta)-\lambda_o\,\theta\,(R\,d\theta)=\lambda_o\,\pi R\,d\theta=(Q/2\pi)d\theta. \quad (1)$$

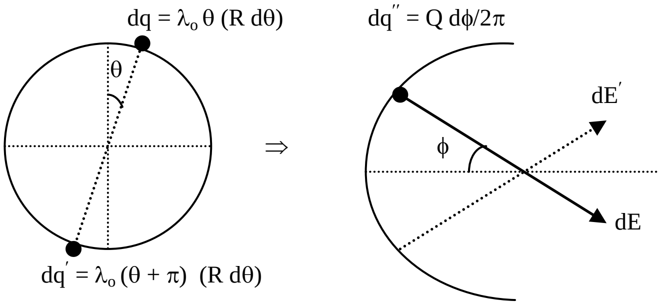

FIGURE 1. Equivalent charge distribution for the circular wire with $\lambda=\lambda_o\,\theta$.

Thus, the circular wire with $\lambda=\lambda_o\,\theta$ is equivalent to a semicircular wire of constant charge density $Q/2\pi R$. Again, we exploit symmetry, by splitting the region into the upper quarter and the lower quarter and consider the contribution to the electric field due to symmetrically placed charge elements. The vertical components cancel out while the horizontal components add up. Thus, the direction of **E** follows directly from symmetry, without any calculation! Using integration, it is shown easily that $E=kQ/\pi R^2=0.3183kQ/R^2$, where $k=1/4\pi\varepsilon_o$.

What about those students who are unfamiliar with calculus /integration? For them, the explanation proceeds as follows: The circular ring may be split into many small point charges. Adding up the contribution due to all of these, we get **E**. To start with, let us split the ring into n = 6 equal arcs and replace each arc with a point charge at its centre. Thus, the first arc between 0^o and 60^o is replaced by a point charge q at 30^o. The next arc between 60^o and 120^o is replaced by a point charge at 90^o. Since the charge distribution is proportional to the angle, the charge at 90^o is 3q, three times the charge at 30^o; etc. Thus, we get the situation in Fig. 2.

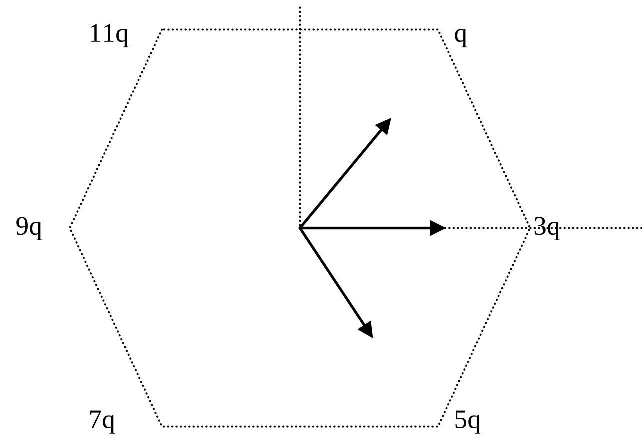

FIGURE 2. Equivalent point charges for the circular wire with $\lambda=\lambda_o\,\theta$.

Considering the cancellations for **E** at the centre from pairs of oppositely placed charges, we have only 3 contributions to **E**, as shown, of equal magnitude. Thus, the direction of **E** is obtained without any calculation! A simple calculation gives the magnitude of **E** as $0.3333kQ/R^2$. Comparing this with the value obtained from integration, we see an error of 4.7%, which is not too bad, considering the approximations we have made! When we approximate the circular ring with n=12 point charges and proceed as above, we obtain the value of $0.3220kQ/R^2$ for E, which has an error of only 1.2%. Incidentally, this also proves to the students that the accuracy increases with n. But then, they may complain that bigger n means more work. That gives us an excellent opportunity to entice them to learn integration, where $n=\infty$.

B. A Straight Wire with Linear Increase in λ

Consider a straight wire of length 2L and total charge +Q, with the charge density increasing linearly from zero on the left side. Our aim is to calculate the magnitude and direction of the net electric field **E** at a perpendicular distance "a" from the center of the wire.

Let us write the given density variation as:

$$\lambda = \lambda_o\,x',\ 0 \le x' \le 2L. \quad (2)$$

Using simple integration, we can show that

$$\lambda_o = Q/2L^2. \quad (3)$$

At first, the problem seems to have no symmetry at all. To get a better insight, it is helpful to shift the origin of the X-axis to the midpoint of the wire, using the relation

$$x' = x + L. \quad (4)$$

Using this, we see that

$$\lambda = \lambda_o\, x + \lambda_o\, L. \quad (5)$$

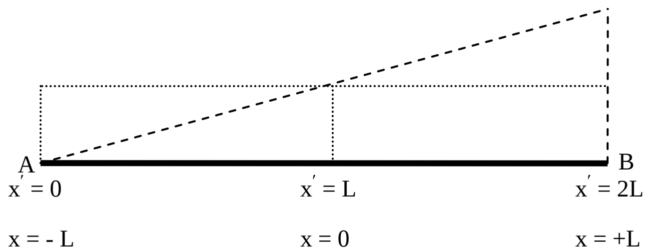

FIGURE 3. Variation of the charge density in x′ & x frames.

Thus, using the new expression for λ (or, simply from Fig. 3), we see that the given wire is equivalent to the sum of a first wire of length 2L, with uniform charge density $\lambda_1=\lambda_o L$ plus a second wire of length 2L, with charge density $\lambda_2=\lambda_o x$, RHS being +ve and LHS being -ve. We see that the first wire has an even symmetry, since $\lambda(-x)=\lambda(+x)$, while the second wire has an odd symmetry, since $\lambda(-x)=-\lambda(+x)$. From Fig. 3, or from simple calculation, we can show that the charge on the first wire $Q_1=Q$, while for the second wire, Q_2 (RHS)$=Q/4=-Q_2$ (LHS).

C. The First Wire with Uniform λ

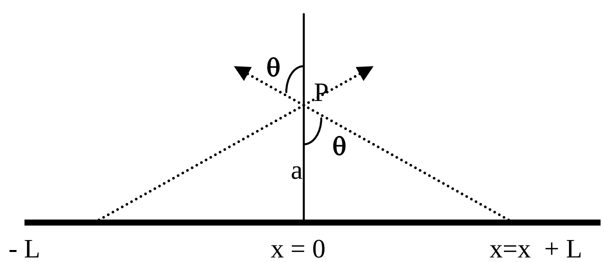

FIGURE 4. E due to the first wire, with uniform charge density.

Fig. 4 shows the first wire with uniform charge density λ_1. The charge in a small element dx at x is $dq=\lambda_1\, dx$. Due to it, at P:

$$\overrightarrow{dE} = \frac{k\,\lambda_1\, dx}{x^2+a^2}, \ \ \textit{as shown} \quad (6)$$

From symmetry, we see that the horizontal components cancel out, while the vertical components add up.

$$dE_v = dE\,\cos\theta = \frac{k\,\lambda_1\, a\, dx}{\left(x^2+a^2\right)^{3/2}}. \quad (7)$$

Thus,

$$\vec{E}=E_v=2k\,\lambda_1 a\int_0^L \frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{kQ}{a\sqrt{a^2+L^2}}. \quad (8)$$

Now, for the students who have not done calculus, we proceed as follows, so that they too can enjoy the beauty and power of symmetry in Physics. For this, we split L into small segments of charge q and length "2l", the charge in each segment being placed at its center. e.g. L = 12 l. In this case, we have six segments of charge q and length "2l" on the RHS, and similarly on the LHS. The situation is then as shown in Fig. 5 below:

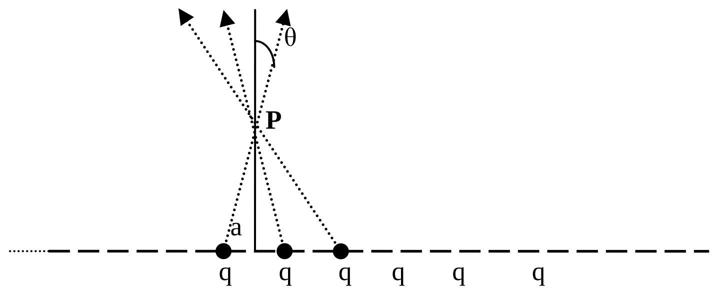

FIGURE 5. Q split into 12 elements uniformly distributed.

Considering the two charges at "+l" and "-l", we see that their horizontal components at P cancel out, while the vertical components add up, so that their contribution is:

$$\vec{E}(+l\,\&-l)=E_l\,(vert)=\frac{2kq}{a^2+l^2}\,\cos\theta_1=\frac{2kqa}{\left(a^2+l^2\right)^{3/2}}. \quad (9)$$

The contribution due to the other 5 pairs is obtained similarly. For comparing the result with the exact result from integration, we need to express L in terms of "a". When L=2a, and L=12 l, the result is $E=E_v=0.447\,4598kQ/a^2$, with a negligible error of 0.06%.

It should be noted that the percentage error depends not only on the number of small segments into which L is split, but also on the value of L/a. When L = 8a, even with L = 16 l, the error is 1.2%. This again shows to the students the advantage of learning calculus.

D. The Second Wire with $\lambda_2=\lambda_o x$, $-L \le x \le +L$

Fig. 6 shows the second wire with $\lambda_2=\lambda_0\, x$. The charge in a small element dx at x is $dq=\lambda_0\, x\, dx$. Due to it, at P:

$$\overrightarrow{dE} = \frac{k\,\lambda_0\, x\, dx}{x^2+a^2}, \quad (10)$$

where, using Eq. (3),

$$\lambda_0 = Q/2L^2 = 2\,Q_2/L^2, \quad (11)$$

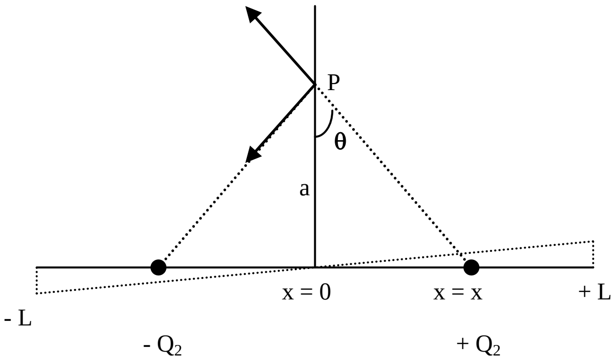

FIGURE 6. E due to the second wire, with $\lambda_2=\lambda_0$ x.

since $Q=4Q_2$, as mentioned above. From symmetry, we see that the vertical components cancel out, while the horizontal components add up. From symmetry, we see that the vertical components cancel out, while the horizontal components add up, so that

$$dE_h = dE\sin\theta = \frac{2kQ_2}{L^2}\,\frac{x^2\,dx}{\left(x^2+a^2\right)^{3/2}}, \qquad (12)$$

$$\vec{E}=E_h=2\int_0^L dE_h=\frac{4kQ_2}{L^2}\left[\ln\left(L+\sqrt{L^2+a^2}\right)-\ln a-\frac{L}{\sqrt{L^2+a^2}}\right]. \qquad (13)$$

In solving the problem without integration, we split the wire into many small segments of width "2l" and place the charge in each segment at the center of that segment. The charge q in the first segment (between 0 & 2l) is placed at x=l. The charge in the next segment (between 2l & 4l) is placed at x=3l. Since, the charge density increases linearly with distance, the charge element at x=3l is 3q. Thus we get the situation in Fig. 7 below:

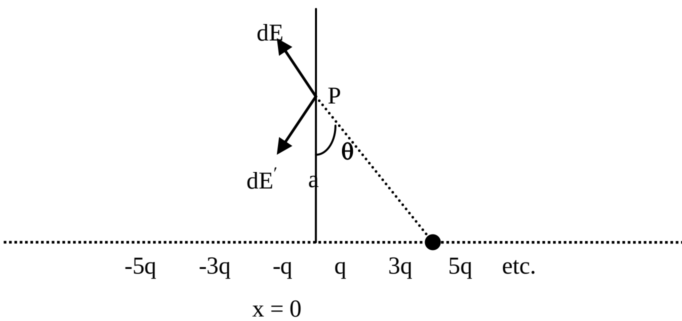

FIGURE 7. Point charges due to the second wire, with $\lambda_2=\lambda_0$ x.

Consider the charges "+nq" & "-nq" at "+nl" & "-nl" contributing d**E** & d**E**′. Their vertical components at P cancel out, while their horizontal components add up. Hence

$$\vec{E}(+nq\,\&\,-nq)=E_h=\frac{2knq}{a^2+n^2l^2}\sin\theta = 2kql\,\frac{n^2}{\left(a^2+n^2l^2\right)^{3/2}} \qquad (14)$$

Summing this up over the various pairs, we obtain the net E. As mentioned in the earlier sections, the accuracy increases with increasing value of n_{max}.

V. DISCUSSIONS & CONCLUSIONS

Thus, with carefully chosen problems, we can communicate to our students the beauty and power of symmetry in their problem-solving. Put this way, we will help some of our students to realize that solving a Physics problem could be like a jigsaw puzzle – an art work! Searching for truth (in Physics) does not have to be a sterile job. There are elements of beauty, discovery and excitement too! As Poincare put it, "If nature were not beautiful, it would not be worth knowing and life would not be worth living".

"Let's make sure that when the students think back to our classes, they remember more than the struggle with equations. Let's do our best to see to it that they also remember the wonder, the connections, the excitement of discovery and the poetry of the universe" [5].

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