Fourier heat transfer and the piston speed

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Abstract

We point out that, for an ideal gas contained in a cylinder and interacting with a reservoir isothermally, the concepts of heat transfer rate *q(t)* and piston's velocity *u(t)* are strongly inter-related. If the heat exchangers enforce Fourier's law *q*=constant the resulting *u(t)* increases with the time *t* exponentially leading to possible turbulence in the gas. Furthermore, if coupling to a crank shaft enforces *u*=constant then the resulting *q(t)* violates Fourier's law at large times.

Keywords: Heat transfer, Thermodynamics.

Resumen

Señalamos que, para un gas ideal contenido en un cilindro e interactuando isotérmicamente con un depósito, los conceptos de tasa de transferencia de calor *q(t)* y la velocidad de pistón *u(t)* están fuertemente interrelacionados. Si los intercambiadores de calor hacer cumplir la ley de Fourier *q* = constante, la resultante *u(t)* aumenta con el tiempo *t* exponencialmente para posibles turbulencias en el gas. Además, si el acoplamiento a un cigüeñal aplica *u* = constante entonces el resultado *q(t)* viola la ley de Fourier en tiempos grandes.

Palabras clave: Transferencia de calor, Termodinámica.

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I. INTRODUCCIÓN

For the last three decades we have been teaching the pioneering work of Curzon and Ahlborn [1] on the efficiency of an endoreversible Carnot engine at maximum power output in our undergraduate thermodynamics class. Although the students were very impressed by the elegance of CA's derivation yet some inquisitive ones asked as to what the piston's speed would be on the isothermal branches of the cycle assuming Fourier's law of heat transfer. Of course, no such question was raised about the adiabatic branches because those were supposed to be instantaneous any way in the CA approach. The purpose of this note is to illuminate the answer to the above question for the benefit of students, teachers and researchers associated with the development of finite time thermodynamics (FTT) [2, 3, 4, 5, 6, 7].

II. FORMULATION

A. General Considerations

The system to be considered is one mole of an ideal gas having mass M_g contained in a cylinder fitted with a piston of unit cross section. During a quasi-static thermodynamic process a differential amount of heat *dQ* is absorbed from a reservoir in time *dt* but no assumption at this stage is made about the rate of heat transfer *q(t)* or its integral *Q(t)* defined by

$$q(t) \equiv dQ/dt; Q(t) \equiv \int_{t_i}^{t} dt' q(t') \qquad (1)$$

where the suffix *i* labels the initial state. Also, the instantaneous velocity *u(t)* of the piston and the corresponding volume *V(t)* of the gas are defined by

$$u(t) \equiv dV/dt; V(t) \equiv V_i + \int_{t_i}^{t} dt' u(t') \qquad (2)$$

without making any assumption as to how the piston is coupled to an external load.

B. Use of the First Law

The precise link among the symbols appearing in Eqs. (1) and (2) is provided by the widely known first law [8] of thermodynamics.

$$dQ = dU + PdV \quad (3)$$

Where dU is the differential change of internal energy and P denotes the pressure of the gas. For simplicity let us consider an isothermal expansion occurring at constant temperature T so that the ideal gas equation of state tells

$$dU = 0; P = RT/V = M_g c_s^2 / V; c_s \equiv \left(RT/M_g\right)^{1/2} \quad (4)$$

With c_s being the speed of sound in the isothermal medium. Combining Eqs. (1) and (4) we find $dQ = RTdV/V$ whose integral yields the usual expression for the net heat $Q(t)$ absorbed during the isothermal process

$$Q(t) = RT\ln\left(V/V_i\right) \quad (5)$$

C. Main Results of the Paper

Of greater interest, however, is its time derivative

$$q(t) \equiv dQ/dt = \left(RT/V\right)dV/dt = RTu\Big/\left[V_i + \int_{t_i}^{t} dt' u(t')\right] \quad (6)$$

which is the first main result of our paper. It shows that "if the piston's velocity profile $u(t')$ is pre-specified (e. g., by coupling to a suitable crank shaft) then the corresponding heat transfer rate $q(t)$ gets automatically determined from Eq. (6), e. g., it need not necessarily obey Fourier's law." An example corroborating this statement will be supplied later in Eq. (10)

Next, we can invert Eq. (5) to write the volume as $V(t) = V_i \exp\left(Q/RT\right)$. Its time derivative reads

$$u(t) \equiv dV/dt = \left(V/RT\right)dQ/dt = \left(V_i q/RT\right)\exp\left[\left(1/RT\right)\int_0^t dt' q(t')\right] \quad (7)$$

which is the second main result of our paper. It shows that "if the heat transfer rate $q(t')$ is pre-specified (e. g., by choosing suitable heat exchangers) then the corresponding piston's velocity $u(t)$ gets determined automatically from Eq. (7), *i. e.*, it need not be necessarily quasi-static." An example corroborating this statement will be supplied below in Eq. (8).

III. ILLUSTRATIVE EXAMPLES

A. Fourier's Law of Heat Transfer

It is commonly believed [1] that if a working substance at temperature T is in contact with a reservoir at higher temperature $T_H = T + x$ then the heat transfer rate is governed by the famous Fourier's prescription $q = \alpha x$, $Q(t) = (t-t_i)\alpha x$ where α is the relevant thermal conductance. For constant αx the piston's velocity u is easily computed from our Eq. (7) to be

$$u(t) = u_i \exp\left[(t - t_i)/\tau_0\right] \quad (8)$$

where the characteristic time scale τ_0 and piston's initial velocity u_i are read-off from

$$\tau_0 = RT/\alpha x; u_i = V_i \alpha x/RT = V_i/\tau_0 \quad (9)$$

Eq. (8) is the precise answer to the student's question asked in the introduction. It tells that, even if u_i is quasi-static (i.e., $|u_i| << c_s$) initially, the ensuing speed $|u|$ grows exponentially with the time span $(t - t_i)$ leading to turbulence in the gas at $(t - t_i) >> \tau_0$. Therefore, the assumption of a pure Fourier's law (made frequently in FTT) leads to bad physics at long time span or for large expansion ratio $V(t)/V_i$

B. Piston having constant Velocity

Next, we consider a simple pre-specified model for the piston's movement namely $u = u_i, V(t) = V_i + (t - t_i)u_i$ at all times of interest. The corresponding heat transfer rate q is readily evaluated from our Eq. (6) as

$$q(t) = RTu_i/\left[V_i + (t - t_i)u_i\right] = \alpha\, x/\left[1 + (t - t_i)/\tau_0\right] \quad (10)$$

This relation also shows that q must deviate from Fourier's prescription after a long time span $(t - t_i) >> \tau_0$.

IV. CONCLUDING REMARKS

- It is hoped that students will find interesting our main results in Eqs. (5) and (7). These results have not been given explicitly in the existing FTT literature.
- The basic message to be learnt from our examples in Eqs. (8) and (10) is that the concepts of heat transfer rate and piston's speed are strongly interlinked so that the strict Fourier's law for q should not be pre-assumed on the isothermal branch of the endoreversible Carnot cycle.
- Dynamically the piston's velocity u can be obtained by solving the relevant Newton's law of motion for unit cross section namely $M\, du/dt = P - P_{ext}$ where M is piston's mass and P_{ext} the back force from the load (including friction with the wall of the cylinder).The solution of this equation, however, is a nontrivial task deferred for a future communication.

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