Characteristic dimensions for heat transfer

E. Marín
Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada
Instituto Politécnico Nacional, Legaria 694, Col. Irrigación, C.P. 11500, México D.F., México

Email: emarin63@yahoo.es; emarinm@ipn.mx

(Received 10 November 2009, accepted 16 January 2010)

Resumen

Teniendo en cuenta la necesidad y la importancia de introducir en el currículo de estudio de física e ingeniería los conceptos fundamentales relacionados con la transferencia de calor, en este trabajo, a partir de las soluciones de la ecuación de conducción de calor no estacionaria, discutiremos las escalas espaciales y temporales involucradas en las leyes de ese fenómeno y los límites de validez de su descripción actual, con la esperanza de que esos conceptos puedan ser de utilidad para profesores y estudiantes.

Abstract

Taking into account the actual need and importance of introducing the main concepts of heat transfer in the curriculum of science and engineering studies, and starting from the solutions of the non-stationary heat conduction equation, we discuss the length and time scales involved in the laws of this phenomenon and the limits of validity of its current description, hoping that these concepts can be helpful for teachers and students.

Keywords: Transferencia de calor, dimensiones, fonones

PACS: 01.90.+g, 44.25.+f, 47.20Bp, 47.55.Pb **ISSN 1870-9095**

I. INTRODUCTION

The conduction of heat in solids is a well known phenomenon, whose mathematical description dates from about two hundred years before, when Fourier [1] stated the famous law having his name. However, the role played in these phenomena by the thermal parameters governing the heat transport is often not well known or it is misinterpreted [2]. In particular, there are some characteristic (length and time) dimensions playing a very important role in the understanding of heat transfer phenomena. As there are many size and time dependent physical properties (in other words, these properties can be different for dissimilar phenomena), it is of great importance to deal with this theme with students (and teachers) at a college and university level. Therefore, it is the main objective of this work to define, in a phenomenological and easy accessible way, characteristic lengths and time dimensions for the very important phenomena of macroscopic heat transfer and to discuss about the limits of the laws describing it. This paper is distributed as follows. In the next section the laws of macroscopic heat transfer by conduction will be presented and the physical meaning of the involved thermal parameters will be briefly discussed. Sections III and IV will be devoted to the analysis of characteristic length and time scales respectively, both within the frame of non-stationary heat conduction in the presence of pulsed heat sources. In section V an analysis of the limiting scales for periodical heat sources will be presented. The above aspects will be summarized in section VI together with the conclusions.

II. LAWS OF MACROSCOPIC HEAT TRANSFER BY CONDUCTION

It is well known that any temperature difference within a physical system causes a transfer of heat from the region of higher temperature to the one of lower. This transport process takes place until the system has become uniform temperature throughout. The rate of heat flux (units of W) transferred per unit time, t, depends on the nature of the transport mechanism, which can be radiation, convection or conduction (or a coupling of then) [3]. In this paper we will focus our attention to heat transfer by conduction in condensed matter, for which the local heat flow-rate in some direction, r, of homogeneous material is governed by Fourier's law:

$$\Phi = -kA\nabla T \, . \tag{1}$$

The thermal conductivity, k (W/mK), is expressed as the quantity of heat transmitted per unit time, t, per unit area, A, and per unit temperature gradient $\nabla T=\partial T/\partial \mathbf{r}$. The negative sign indicate that heat flow will take place in the opposite direction of the temperature gradient. It characterizes stationary processes of heat transfer.

When a material is subjected to non-steady heating or cooling, its inner temperature profile is given in terms of time and spatial positions. The resulting conduction can be analyzed by combining Fourier's law with a heat flow balance (energy conservation law or continuity equation). Assuming constant thermal conductivity this leads to a parabolic heat diffusion equation, often called Fourier's second law:

$$\nabla^2 T - \frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t} = 0\,, \tag{2}$$

where ∇^2 is the Laplacian Operator. When internal heat sources are present a term equal to $-Q(r,t)/k$ will be added to the left hand side of this equation, where Q (J/m^3s) denotes possible heat losses (Q<0) or generation (Q>0) per unit volume per unit time. The coefficient α (m^2/s) represents the thermal diffusivity, the rate at which a temperature variation propagates through the material. It can be defined as

$$\alpha = k / C\,, \tag{3}$$

where C ($Jcm^{-3}K^{-1}$) is the specific heat capacity, or heat capacity per unit volume of the material, defined as the product of density, ρ (g/cm^3), and specific heat, c (J/gK):

$$C = \rho\, c\,. \tag{4}$$

Specific heat is the amount of heat that is required to raise the temperature of a unit mass of a substance by one degree, characterizing static problems in heat transfer. Thermal diffusivity can be considered, therefore, as the ratio of heat conducted through the material to the heat stored per unit volume.

Another important thermal parameter for time varying heat transport phenomena is the so called thermal effusivity [$Ws^{1/2}cm^{-2}K^{-1}$]. It is defined as

$$\varepsilon = \sqrt{k\rho c} = {k}\big/{\sqrt{\alpha}} = \rho c\sqrt{\alpha}\,, \tag{5}$$

and it determine the amplitude of the temperature at a sample surface and its behaviour at interfaces in the case of transient and periodical heat sources. A more detailed explanation about the physical mean of the thermal parameters can be found elsewhere [4, 5].

III. CHARACTERISTIC LENGTH

In order to visualize the typical behaviour of a diffusive phenomenon, such as the non-stationary heat conduction described above, it can be useful to start the analysis considering a semi-infinite homogeneous medium experiencing (uniformly, *i.e.* in such a way that the one-dimensional approach used in what follows is valid) a sudden temperature change at the surface at x=0 from T_0 to T_1. For the calculation of the temperature field in the medium created by a heat pulse at t=0 one has to solve the homogeneous heat diffusion equation

$$\frac{\partial T(x,t)}{\partial t} = \alpha\,\frac{\partial^2 T(x,t)}{\partial x^2}\,, \tag{6}$$

with the boundary conditions

$$T(x = 0,\, t \geq 0) = T_1\,;\;\; T(x > 0,\, t=0) = T_{0.} \tag{7}$$

The solution of Eq. (6) with these conditions leads to [6]:

$$T(x,t) = T_1 + (T_0 - T_1)\,erf\left(\frac{x}{2\sqrt{\alpha t}}\right) \text{ for } t \geq 0, \tag{8}$$

where *erf* is the error function defined as

$$erf\,(u) = \frac{2}{\sqrt{\pi}}\int_0^u \exp\left(-\xi^2\right)d\xi\,, \tag{9}$$

so that $erf(0)=0$ and $erf(u\to\infty)=1$ (see Appendix IV of Ref. 6 for other properties, such as the first derivative formula that we will use below).

From the temperature field given by the above equation one may deduce, by differentiation, the heat flow, $q=\Phi/A$, given by the Fourier's law of conduction (Ec. (1)). In our case this lead to

$$q = \frac{(T_1 - T_0)}{\varepsilon\sqrt{\pi\, t}}\exp\left(-\frac{x^2}{4\alpha t}\right) = q_0\exp\left(-\frac{x^2}{{\mu_t}^2}\right). \tag{10}$$

This expression describes a Gaussian spread of thermal energy in space and time with characteristic width

$$\mu_t = 2\sqrt{\alpha t}\,, \tag{11}$$

giving the distance at which the amplitude of the heat flux reduces e times from its sample's surface value at x=0 at each time instant, *i.e.*, the heat flux amplitude q_0. (The demonstration of this assertion would be a good exercise for students which can remember similar calculations in other branches of physics, such as that of the half live times from the law of radioactive decay, or the optical absorption length from the Lambert-Beer Law, among others). This characteristic distance is often denoted as the thermal diffusion length (for pulsed excitation). Scaling this result to three dimensions one can show that after a time t has elapsed the heat outspread over a sphere of radius μ.

IV. CHARACTERISTIC TIMES

Now suppose that a spherical particle of radius R is heated in the form described above by a heat pulse at its surface. The particle requires for thermalize a time equal to the necessary for the heat to diffuse throughout its volume. The heat flux at the opposite surface of the particle can be obtained using Eq. (10), *i.e*.

$$q\left(r=2R\right)=q_0\exp\left(-\left(\frac{2R}{\mu}\right)^2\right)= \\ q_0\exp\left(-\frac{R^2}{\alpha t}\right)=q_0\exp\left(-\frac{\tau_C}{t}\right), \tag{12}$$

where we can see that the thermal time constant

$$\tau_C=\frac{R^2}{\alpha}, \tag{13}$$

(thus characterizing heat transfer rates) depend strongly on particle size and on its thermal diffusivity, α. The same result has been recently reported by Greffet [7] considering a heat pulse with a delta profile and by Wolf [8] from energy balance criteria. However, the solution given by the first mentioned author for the temperature field is questionable since it leads to a heat flux vanishing at the sample surface at x=0, what is physically undesirable.

As for most condensed matter samples the order of magnitude of α is 10^{-6} m^2/s, for a sphere of diameter 1 cm one obtain τ_c=100 s and for a sphere with a radius of 6400 km, such as the Earth, this time is of around 10^{12} years, both values compatible with daily experience. But there is a problem when we deal with spheres having diameters between 100 and 1 nm (they are characteristic dimensions of the so called nanoworld), for which we obtain for these times values ranging from about 10 ns to1 ps. They are very close to the relaxation times, τ, or build-up times necessary for the onset of a heat flux after a temperature gradient is imposed on a given sample. At these short time scales Fourier's Law of heat conduction is not more valid in the form given above.

In effect, something appears paradoxical in the above description because Eq. (1) gives rise to infinite speeds of heat propagation. In other words, if we apply at a given instant a supply of heat to, for example, one face of a flat slab, according to Eq. (1) there will be an instantaneous effect at the rear side, *i.e*., the flux of heat reacts simultaneously to the concentration gradient leading to an unbounded propagation speed. This conclusion, of course, is physically not reasonable. This paradox was resolved in the mid of the past century [9] with the postulation of the so-called modified Fourier's law, also known as Cattaneo's Equation:

$$q(x,t)+\tau\frac{\partial q(x,t)}{\partial t}=-k\frac{\partial T(x,t)}{\partial x}, \tag{14}$$

which tell us that, as a consequence of the temperature existing at each time instant, t, the heat flux appears only in a posterior instant, $t+\tau$. Substituting Eq. (14) in the law of energy conservation [10] one can obtain the hyperbolic heat diffusion equation with first and second order time derivatives of the temperature field [11]:

$$\frac{\partial^2 T(x,t)}{\partial x^2}-\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}-\frac{1}{u^2}\frac{\partial^2 T(x,t)}{\partial t^2}=0, \tag{15}$$

where

$$u=\sqrt{\frac{\alpha}{\tau}}, \tag{16}$$

has a dimension of velocity. Note that if the characteristic time τ_c is much longer than the relaxation time τ, as occur in daily macroscopic phenomena, the second term in the right hand side of Eq. (14) can be neglected and the Fourier law becomes valid, thus the heated system gradually thermalize becoming isothermal and following any temporal variation of the heat flux imposed upon it. But for characteristic times approaching the relaxation time or shorter than it, the system cannot follow the imposed temporal changes and it behaves like a low-pass filter responding only to the mean value of the heat flux. This result can be better explained if we consider heating by an harmonic heat flux on the sample surface, instead of the above analyzed pulsed flux, as we will seen in the following paragraph.

V. CHARACTERISTIC SCALES: HARMONIC HEAT SOURCES

Consider an isotropic homogeneous semi-infinite solid, whose surface is heated uniformly by a source of periodically sinusoidal modulated intensity $I_0(1+\cos(\omega t))/2$, where $\omega=2\pi f$ is the angular modulation frequency, and t is the time. The temperature distribution $T(x,t)$ (without loss of generality we will also consider the one dimensional situation) within the solid can be obtained by solving the homogeneous heat diffusion equation (15) with the boundary condition

$$-k\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0}=\mathrm{Re}\left[\frac{I_o}{2}\exp(i\omega t)\right], \tag{17}$$

which express that the thermal energy generated at the surface of the solid is dissipated into its bulk by diffusion. The solution of interest for the problem described by the above equations can be proposed as:

$$T(x,t)=\mathrm{Re}\left[\Theta(x)\exp(i\omega t)\right]. \tag{18}$$

Substituting in Eq. (15) we obtain

$$\frac{d^2\Theta(x)}{dx^2}-{q_c}^2\Theta(x)=0, \tag{19}$$

where

$$q_c^2 = \frac{i\omega}{\alpha}\left(1 + i\frac{\omega}{\omega_l}\right), \tag{20}$$

and

$$\omega_l = \frac{1}{\tau}. \tag{21}$$

Two limiting cases can be examined.

First, for $\omega << \omega_l$ we have

$$q_c = \sqrt{\frac{i\omega}{\alpha}} = (1+i)\sqrt{\frac{\omega}{2\alpha}} = \frac{(1+i)}{\mu}, \tag{22}$$

where

$$\mu = \sqrt{\frac{2\alpha}{\omega}}. \tag{23}$$

The general solution of Eq. (19) has in this case the form

$$\Theta(x) = \frac{I_o}{2\varepsilon\sqrt{\omega}}\exp\left(-\frac{x}{\mu}\right)\exp\left[-i\left(\frac{x}{\mu} + \frac{\pi}{4}\right)\right], \tag{24}$$

describing a well known attenuated plane thermal wave whose characteristics have been described in detail elsewhere [12] and which play a very important role in the very exciting field of photothermal phenomena and techniques [5, 12]. This is the same solution one obtains when solving the parabolic heat diffusion, as one can demonstrate in a straightforward manner by substituting Eq. (18) in Eq. (2). In this case one obtains also for the wave number the result given by Eq. (22). More details can be found for example in our recent published work [5]. Thus, when thermal waves are excited by intensity modulated heat sources we may consider that for low frequencies such that $\omega << \omega_l$ the parabolic approach is valid. This case represents a mode similar to that described above for $\tau_c >> \tau$, through which the heat generated in the sample is transferred to the surrounding media by diffusion at a rate determined by the thermal diffusivity. The thermal diffusion length (for periodical excitation), μ, gives the distance at which an appreciable energy transfer takes place and it is known also as a thermal diffusion length, with a physical significance similar to that of μ_t in time domain.

It is worth to notice that the time constant τ in condensed mater is related to the phonon relaxation time, which is in the picosecond range, so that the limiting frequency becomes about $\omega_l = 10^{12}$Hz. As typical modulation frequencies in photothermal techniques are far beyond this value, then the use of the parabolic heat diffusion equation is sufficient for the analysis of the experimental results, what is advantageous when characterizing systems of small dimensions such as the recent developed colloidal suspensions of nanoparticles, the so called nanofluids, as described recently [13].

On the other hand, for $\omega >> \omega_l$, the wave number becomes

$$q_c = i\frac{\omega}{u}, \tag{25}$$

and the solution of Eq. (19) is

$$\theta(x) = \frac{I_0 u}{2k\omega}\exp\left[-i\left(\frac{x\omega}{u} + \frac{\pi}{2}\right)\right], \tag{26}$$

i.e., an harmonic thermal wave traveling at a given frequency without attenuation and with the velocity u. This case, discussed in more detail elsewhere [12] represents a form of heat transfer which takes place through a direct coupling of vibrational modes (i.e. the acoustic phonon spectrum) of the material. The velocity u is then related to the velocity of propagation of the heat carriers, *i.e.*, to the velocity of sound in a medium. At these high frequencies (short time scale) ballistic transport of heat can be dominant and of importance at the nanoscale. In other words, for $\tau_c << \tau = 1/\omega$ and/or $\omega >> \omega_l = 1/\tau$ the media becomes non-dispersive for the heat flux, analogous to what happens with electromagnetic waves in the case of a non-dispersive medium, *i.e.* a medium with negligible electrical conductivity as a vacuum (for electromagnetic waves in an electrical conductive medium such as a metal one can define the skin depth, analogous to the thermal diffusion length).

Characteristic lengths are then of the order of $\Lambda_m = u\tau$, *i.e.*, of the mean free path of the phonons, the average distance a phonon travels between successive collisions. This determines the other important characteristic dimension when dealing witch heat transport phenomena and it is involved in the transition between diffusive and ballistic heat transport regimes. If the mean free path is much shorter than the characteristic length, the Fourier law is valid locally. If not, non-local effects must be taken into account. It is worth to notice that when the dimensions of the system are comparable with the phonon wave length the heat transfer can be quantized and a quantum of thermal conductance can be defined, representing the maximum possible value of energy that can be transported by phonon mode at a given temperature, as has been discussed by Schwab *et al* [14]. This is a very interesting and actual theme that is behind the scope of the present work.

On the other hand, quantum mechanics shows that the mean free path is a function of the phonon frequency [15]. Low-frequency phonons have large mean free paths and vice versa. At high temperature one defines an average mean free path. Therefore, as discussed by Cahill *et al* [16] and more recently by Wautelet and Duvivier in the framework of nanoscale thermal transport [17], the temperature, the basic parameter of Thermodynamics, may not be defined at very short length scales but only over a length larger than Λ_m since its concept is related to the average energy of a system of particles. ¿What is then the size of the regions over which a local temperature can be defined? The above mentioned authors state that as the mean free path is in the nanometer range for many materials at room temperature, systems with characteristic dimensions below about 10 nm are in a non-thermodynamical regime, although the concepts of thermodynamics are often used in the description of heat transport in systems with these dimensions. To the author knowledge there is no a comprehensible and well established way yet to solve this very important problem about the definition of temperature in such systems.

The discussed characteristic scale parameters are summarized in Table I.

TABLE I. Characteristic dimensions in heat transfer.

Parameter	Brief description
R	Particle radius or typical dimension of the physical system
$\mu_t = 2\sqrt{\alpha t}$	Effective thermal diffusion length (in the time domain, for pulsed heating)
$\tau_C = \dfrac{R^2}{\alpha}$	Thermal time constant for a system of dimension R
$\mu = \sqrt{\dfrac{2\alpha}{\omega}}$	Thermal diffusion length (in frequency domain, for periodical heating)
τ	(Phonon) relaxation time or build on time for the heat flux after a temperature gradient is imposed on a sample.
Λ_m	Phonon mean free path
$\omega_l = 1 / \tau$	Cut off frequency between diffusive and ballistic heat diffusion regimes in the case of harmonic heat sources.

VI. CONCLUDING REMARKS

One can conclude by analyzing the characteristic time and length scale parameters of a physical system showed above, that the limits of the macroscopic approach to heat transfer appear when they become comparable with the phonons relaxation times and mean free path, respectively. Controversial aspects of the heat transfer at reduced dimensions have been analyzed in the last years by several authors. However, no definitive, clear explanation of the involved physical mechanisms has been offered so far. But the phenomenological aspects described here suggests the possibility of dealing in advanced or introductory physics courses with concepts related to heat transfer applied to systems of reduced dimensions, which appear in the growing fields of nanoscience and nanotechnology, moletronics, mesoscopic systems, among others. We feel that the approach presented in this study will aid in opening the literature associated with this theme to teachers and students.

ACKNOWLEDGEMENTS

This work was partially supported by SIP-IPN (Project # 20090160), COFAA-IPN (SIBE program) and CONACyT (Grant Nr. 83289), México.

REFERENCES

[1] Fourier, J. B., *The Analytical theory of Heat*, (Encyclopedia Britannica, Inc., Chicago, 1952).

[2] Marin, E., *Thermal Physics concepts: The role of thermal effusivity*, The Physics Teacher **44**, 432 (2006).

[3] Madelung, O. and White, G. K. (eds), *Numerical Data and Functional Relationships in Science and Technology*: New Series III, 2.3.1.11, Landolt-Bornstein (2005), *Thermal Transport* (Springer, Berlin, 2005) pp. 2-217.

[4] Salazar, A., *On thermal diffusivity,* Eur. J Phys. **24,** 351 (2003).

[5] Marin, E., *The role of thermal properties in periodic time-varying phenomena*, Eur. J Phys. **28** 429 (2007).

[6] Carlslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids* (Oxford Univ. Press, London, 1959).

[7] Voz, S., (Ed) *Topics in Applied Physics*, edited by. (Springer, Paris, 2007) pp. 1-13.

[8] Wolf, E L, *Nanophysics and Nanotechnology: An Introduction to Modern Concepts in Nanoscience,* (Wiley-VCH, Weinheim, 2004).

[9] Cattaneo, O. C., *Sulla conduzione de calore*, Atti del Semin. Mat. e Fis. Univ. Modena **3**, 83 (1948).

[10] Marin, E., Marin, J. and Hechavarría, R., *Hyperbolic heat diffusion in photothermal experiments with solids*, J. de Physique IV **125**, 365 (2005).

[11] For a review see: Joseph, D. D. and Preziosi, L., *Heat waves*, *Rev. Mod. Phys.* **61**, 41 (1989). See also: Joseph, D. D. and Preziosi, L., *Addendum to the paper "Heat waves"*, *Rev. Mod. Phys.* **62**, 375 (1990).

[12] E Marin, Marin, J. and Díaz, P., *On the wave treatment of the conduction of heat in solids,* Eur. J. Phys. **23**, 523 (2002).

[13] Marín, E., *On the role of photothermal techniques for the thermal characterization of nanofluids,* Internet Electron. J. Nanoc. Moletrón. **5**, 1007 (2007).

[14] Schwab, K., Henriksen, E. A., Worlock, J. M. and Roukes, M. L., *Measurement of the quantum of thermal conductance*, Nature **404**, 974 (2000).

[15] Hermann, R. P. and Grandjean, F., *Einstein oscillators that impede thermal transport,* Am. J. Phys. **73**, 110 (2005).

[16] Cahill, D. G., Ford, W. K., Goodson, K. E., Mahan, G., Majumdar, D. A., Maris, H. J., Merlin, R. and Phillpot, S. R., *Nanoscale thermal transport,* J. Appl. Phys. **93**, 793 (2003).

[17] Wautelet, M. and Duvivier, D., *The characteristic dimensions of the nanoworld*, Eur. J. Phys. **28**, 953 (2007).