Generation of EAS from power law and non-power law potential in D dimensional spaces by transformation method

N. Saikia[1], S. A. S. Ahmed[2]
[1]*Department of Physics, Chaiduar College, Gohpur-784168, India.*
[2]*Department of Physics, Gauhati University, Guwahati -781014, India.*

E-mail: n_saikia06@yahoo.com.in; sasa_gup@hotmail.com

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Abstract

Construction exactly solvable power law and non-power law potential in one arrangement, within the framework of Green's functions technique. We have applied Extended transformation method to generate a set of exactly solved quantum system in any chosen dimensional spaces. The bound state and scattering state solution of the Green's functions equation of the generated quantum system are reported. The new quantum systems are in general, energy dependent with a single normalized eigenfunctions. A method has been devised to convert a subset of the generated quantum systems with energy dependent potential to a single normal system with an energy independent potential. As the non-polynomial (parent) potential is a combination of power law and non-power law potentials, the power law part generates a potential quantitatively similar to the Coulomb+Screened Coulomb potential and the generated potential from the non-power law part identified as Shifted harmonic Oscillator.

Keywords: Schrodinger Green's function equation, Extended transformation method, non-polynomial potential, Screened Coulomb potential, Shifted Harmonic Oscillator potential.

Resumen

En el marco de la técnica de las funciones de Green, se puede hacer la construcción de un potencial polinomial y no polinomial en un arreglo. Hemos aplicado el método de transformación extendido para generar un conjunto de sistema cuántico resuelto exactamente en cualesquier espacios tridimensionales escogidos. Las soluciones generadas de la función de Green del estado base y del dispersado de la ecuación de estado del sistema cuántico son reportadas. Los nuevos sistemas cuánticos son, en general, son dependientes de la energía con funciones propias normalizadas sencillas. Se ha diseñado un método para convertir un subconjunto de los sistemas cuánticos generado con un potencial dependiente de la energía a un sistema normal sencillo, con un potencial independiente de la energía. Ya que el potencial no polinómico (padre) es una combinación de potenciales polinómico y no polinómico, la parte polinómica genera un potencial cuantitativamente similar al del potencial de Coulomb +apantallado y el potencial generado por la parte no polinómica es identificado como oscilador armónico modificado

Palabras clave: Ecuación de función de Green, método de transformación extendida, potencial no polinómico, potencial de Coulomb apantallado, potencial modificado del Oscilador Armónico.

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I. INTRODUCTION

Exact analytic solution (EAS) of Schrodinger equation for a given physical quantum system (QS) is desirable as it conveys maximum information of the system. Considerable effort has been made in recent years towards obtaining the EASs of the Schrodinger equation for certain power law and non-power law potentials of physical interest in lower dimensional spaces [1, 2, 3, 4, 5, 6]. These EASs serve as benchmark for testing the accuracy of other non-perturbative methods. Success in one and three dimensional EAS, can be termed as only a few states can be found analytically. In this succinct report we present a transformation method within the frame work of Green's function (GF) technique [7, 8, 9]. The transformation method is called the Extended transformation (ET), that includes a coordinate transformation (CT) and a functional transformation (FT). As shown in the paper [10, 11] the dimension of the (Euclidean) space into which the transformed system gets transported to, in the case of power law potentials can be arbitrarily pre-assigned, only CT is enough to get standard Schrodinger GF equation form. On the other hand, application of CT alone on the non-power law potentials, we cannot get standard Schrodinger GF equation form. It is therefore imperative to apply ET in case of non-power law potentials.

In this paper we apply ET method to exactly solved 3-dimensional quantum mechanical problem with central non-polynomial potential of Bose [12]

$$V_A(r) = r^2 + \frac{\lambda r^2}{1+\sigma r^2}, \quad (1)$$

λ and σ are the parameter of the potential. Which is a two term potential, first term specifies a power law potential and the second term as a typical representative of non-power law potential. ET method is applied by selecting the working potential (WP) differently from multiterm potential to generates a spectrum of different solved quantum system, each equipped with normalized exact analytic solution and associated energy eigenvalue, in D dimensional spaces. The selection of WP can be made $2^q - 1$ different ways when the original quantum system is a q-term potential. In fact, we have a number of choices to select the WP and the least being a single term. A very useful property of the ET method is that the wave functions of the generated QSs are almost always normalizable.

In the present paper, our main objective is to generate solvable potentials and to show their hierachal connections, since exactly solvable potentials facilitate physical applicability's. A major complication arises that, the generated QSs are almost always sturmian type. We discuss a procedure to regroup this set of energy-dependent sturmian QS to normal QS. The normal QS, called B-quantum system (B-QS) is found to be similar potential of Screened Coulomb potential (when we take r^2 as working potential (WP)) and the non-power law part will generate C-QS, which is identified as the Shifted Harmonic oscillator potential.

The GF equation for the Central non polynomial potential $V_A(r)$, henceforth called the A-system in D_A-dimensional Euclidean space is ($\hbar = 1 = 2m$)

$$\left[\partial_r^2 + E_A - \left(r^2 + \frac{\lambda r^2}{1+\sigma r^2}\right) - \frac{l_A(l_A+1)}{r^2}\right] G_{l_A}(r, r_0, E_A, V_A(r), C_A) = \frac{\delta(r-r_0)}{r_0^{D_A-1}}. \quad (2)$$

Where the GF $G_{l_A}(r, r_0, E_A, V_A(r), C_A)$ and E_A are known for the given $V_A(r)$.

The completeness of the set of energy eigenfunctions allows us to have eigenfunction expansion of GF is

$$G_{l_A}(r, r_0, E_n^A, V_A(r), C_A) = \sum_{n=0}^{\infty} \frac{\varphi_{l_A}^{(n)}(r)\varphi_{l_A}^{*(n)}(r_0)}{E - E_n^A - i\epsilon}. \quad (3)$$

The energy eigenfunctions are

$$\varphi_{l_A}(r) = (a_0 + a_1 r + a_2 r^2 \ldots\ldots\ldots a_m r^m) r^{l_A} e^{-\frac{1}{2}r^2}. \quad (4)$$

Energy eigenvalues are

$$E_A = \frac{\lambda}{\sigma} + 4m + 2l_A + 3. \quad (5)$$

II. CONSTRUCTION OF EAS FOR POWER LAW POTENTIAL

Now we are applying ET to equation (2) which comprises of $r \to g_B(r), r_0 \to g_B(r_0)$ and

$$G_{l_B}(r, r_0, E_N^B, V_B(r), C_B) = f_B^{-1}(r) G_{l_A}\big(g_B(r), g_B(r_0), g_B'^2 E_n^A, g_B'^2 V_A(g_B(r)), C_A\big) f_B^{-1}(r_0). \quad (6)$$

Leads to the following equation:

$$\left[\partial_r^2 + \frac{D_B-1}{r}\partial_r + g_B'^2 \left\{E_A - g_B^2 - \frac{\lambda g_B^2}{1+\sigma g_B^2} - \frac{\left(l_A+\frac{1}{2}\right)^2}{g_B^2} - \frac{1}{4 g_B^2(r)}\right\} + \frac{1}{2}\{g_B, r\} + \frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}\right] G_{l_B}(r, r_0, E_B, V_B(r), C_B) = \frac{\delta(r-r_0)}{r_0^{D_B-1}}, \quad (7)$$

where $g_B(r)$ and $g_B(r_0)$ are the transformation function, which are at least three times differentiable function and $f_B(r)$ and $f_B(r_0)$ are modulated amplitudes. The dimension Euclidean spaces of the transformed quantum system, henceforth called the B-quantum system(B-QS) can be chosen arbitrarily, let it be denoted by D_B. Then $\frac{d}{dr} ln \frac{f_B^2(r) g_B^{D_A-1}(r)}{g_B'(r)} = \frac{D_B-1}{r}$, which fixes $f_B(r)$ as a function of $g_B(r)$ and its derivative,

$$f_B^{-1}(r) = g_B'^{-\frac{1}{2}} g_B^{D_A-1}(r) r^{-\left(\frac{D_B-1}{2}\right)}, \quad (8)$$

$\{g_B, r\} = \frac{g_B'''}{g_B'} - \frac{3}{2}\left(\frac{g_B''}{g_B'}\right)^2$ is the Schwartzian derivative symbol in equation (7) and the quantities $\frac{1}{2}\{g_B, r\} - \frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2} + \left(l_A + \frac{1}{2}\right)^2 \frac{g_B'^2}{g_B^2}$, give the correct form of the centrifugal barrier term in D_B dimensional space $\frac{l_B(l_B+D_B-2)}{r^2}$ (Louk, 1960) [13], whenever $V_A(r)$ is of power law type.

In order to reduce equation (7) to the standard Schrodinger GF equation form the following ansatz have to be made, which are an integral part of the transformation method:

$$g_B'^2 V_A\big(g_B(r)\big) = -E_{N,}^B \qquad (9)$$

The $V_A\big(g_B(r)\big)$, in equation (9) is termed as the working potential (WP) which specifies the transformation function $g_B(r)$. The A-QS potential is a two term potential and we have three (2^2-1) choices to select the WP. The other ansatzes are

$$g_B'^2(r)E_n^A = -V_B^{(1)}(r)\,, \qquad (10)$$

$$g_B'^2(r)\left(V_A\big(g_B(r)\big) - V_A^{(w)}(r)\right) = V_B^{(2)}(r). \qquad (11)$$

Where the B-QS potential $V_B(r) = V_B^{(1)}(r) + V_B^{(2)}(r)$. And

$$\frac{g_B'^2(r)\left(l_A + \frac{D_A}{2} - 1\right)^2}{g_B^2} = \frac{\left(l_B + \frac{D_B}{2} - 1\right)^2}{r^2}. \qquad (12)$$

Now we find different exact analytic solution from equation (4) by setting $m = 0,1,2 \ldots\ldots$

For $= 0$ $\varphi_{l_A}(r) = a_0 r^{l_A} e^{-\frac{1}{2}r^2}$, $E_A = \frac{\lambda}{\sigma} + 2l_A + 3$ and constraint equation for the parameters of the potential is $\frac{\lambda}{\sigma} = 0$. After ET it will generate Coulomb QS.

For $m = 1$, $E_A = \frac{\lambda}{\sigma} + 2l_A + 7$, constraint equation for the parameters and angular momentum quantum number l_A is

$$\left(\frac{\lambda}{\sigma} + 4\right)\left\{\frac{\lambda}{\sigma} + 2\sigma(2l_A + 3)\right\} - 8\sigma(2l_A + 3) = 0\,, \qquad (13)$$

and exact eigenfunctions is given by equation (4) as $\varphi_{l_A}(r) = (a_0 + a_1 r^{l_A})e^{-\frac{1}{2}r^2}$

Selecting $g_B^2(r)$ as the WP from equation (1) and utilizing the ansatze with a simple integration yields:

$$g_B(r) = \pm\sqrt{2r}(-E_B)^{1/4} + C. \qquad (14)$$

The integration constant C is put equal to zero, which attribute the local property $g_B(0) = 0$ which also attribute asymptotic property $g_B(\infty) = \infty$ and we are consider the positive sign in equation (11). Utilizing the ansatz (10), with equation (14) will give

$$V_B^{(1)}(r) = \frac{1}{2}(-E_A)(-E_B)^{1/2}\frac{1}{r} = \frac{C_B^2}{r}\,, \qquad (15)$$

C_B is the characteristic constant of B-QS.

$$C_B^2 = \tfrac{1}{2}(-E_A)(-E_B)^{1/2} = P\ (say), \qquad (16)$$

which will give the energy eigenvalue of B-QS. From equation (11)

$$V_B^{(2)}(r) = \frac{Q}{1 + 2Rr}\,. \qquad (17)$$

The multi-term potential of the B-Sturmian quantum system (B-SQS), given by equation (15) and (17) becomes

$$V_B(r) = \frac{P}{r} + \frac{Q}{1 + 2Rr}\,. \qquad (18)$$

Where $Q = -\lambda E_B$ and $R = \sigma(-E_B)^{1/2}$. The potential is a Sturmian potential as Q and R are n dependent. To make normal QS we require Q and R as n independent constants. These will be achieved only if $\lambda \to \lambda_n$ and $\sigma \to \sigma_n$ such that $\lambda_n = -\frac{Q}{E_B}$ and $\sigma_n = R(-E_B)^{-1/2}$ .The generated B-QS potential is identified as combination of Coulomb and Screened Coulomb potential. The relation between angular momentum quantum number l_A in A-QS and l_B in B-QS is given by equation (12), by equation (14) it can be reduce to

$$2l_A = 4l_B + 2D_B - 5\,. \qquad (19)$$

The parameters and angular momentum quantum number of B-QS are connected by the constraint equation as:

$$\frac{Q^2}{R^2} + 4Q(2l_B + D_B - 1) - \frac{4Q}{R}(-E_B)^{1/2} = 0. \quad (20)$$

Energy eigenvalues of B-QS are given by equation (15) as

$$E_B = -\left[-\frac{Q}{4R} + R(2l_B + D_B - 1)\right]^2. \qquad (21)$$

Thus the transformed B-QS radial GF equation in D_B dimensional Euclidean space is

$$\left[\partial_r^2 + \frac{D_B - 1}{r}\partial_r + E_B - \left(\frac{P}{r} + \frac{Q}{1 + 2Rr}\right) - \frac{l_B(l_B + D_B - 2)}{r^2}\right] G_{l_B}(r, r_0, E_B, V_B(r), C_B) = \frac{\delta(r - r_0)}{r_0^{D_B - 1}}. \qquad (22)$$

Utilizing equation (3) and (6) one obtain the eigenfunctions expansion in terms of GF of B-QS as

$$G_{l_B}(r, r_0, E_B, V_B(r), C_B) = \sum_{N=0}^{\infty} \frac{r^{l_B}(1 + 2Rr)exp[-(-E_B)^{1/2}r]r_0^{l_B}(1 + 2Rr_0)exp[-(-E_B)^{1/2}r_0]}{E + \left[-\frac{Q}{4R} + R(2l_B + D_B - 1)\right]^2 - i\epsilon} \qquad (23)$$

The eigenfunctions can be read off from equation (23) as:

$$\varphi_{l_B}(r) = N_B r^{l_B}(1 + 2Rr)exp\big[-(-E_B)^{1/2}r\big]. \qquad (24)$$

The 2-D view of the potential, energy eigenvalues and eigenfunctions of B-QS are given in Fig.1.

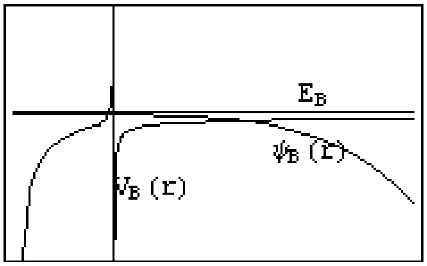

FIGURE. 1. Continuous curves are for $V_B(r) = -\frac{8.5}{r} + \frac{1}{1-2r}$, $\varphi_{l_B}(r) = r(1-2r)exp[1.5r]$ and $E_B = -3.06$. The parameter sets are $P = -8.5$, $Q = 1$, $R = -1$, and $l_B = 1$.

In order to find scattering states we are consider r to be so large that $V_B(r)$ and l_B term in equation (22) can be neglected and write the analogue of equation (23) as:

$$G_B(r, r_0, k^2, V_B(r), C_B) = \int_0^\infty \frac{dE}{E - k^2 - i\epsilon} r(1 + 2Rr)exp[-ikr]r_0(1 + 2Rr_0)exp[-ikr_0]. \quad (25)$$

Using symbolic identity $\lim_{r\to\infty} \frac{1}{(E-k^2)\pm i\epsilon} = P\frac{1}{E-k^2} \mp i\pi\delta(E-k^2)$ we can define imaginary part of $G_B(r, r_0, k^2, V_B(r), C_B)$ and the scattering wavefunction is read as

$$\varphi_B^{scatt}(r) = N_B r(1 + 2Rr)exp[-ikr]\,. \quad (26)$$

III. CONSTRUCTION OF EAS FOR NON-POWER LAW POTENTIAL

Utilizing the power law potentials, for non-power low potentials exact solutions are available only for s-wave, where ($l = 0$). The GF equation (2) now becomes

$$\left[\partial_r^2 + E_A - \left(r^2 + \frac{\lambda r^2}{1+\sigma r^2}\right)\right] G_{l_A}(r, r_0, E_A, V_A(r), C_A) = \frac{\delta(r - r_0)}{r_0^{D_A-1}} \quad (27)$$

Applying ET on equation (27) we obtain

$$\left[\partial_r^2 + \frac{D_C - 1}{r}\partial_r + g_C'^2\left\{E_A - g_C^2 - \frac{\lambda g_C^2}{1+\sigma g_C^2}\right\} + \frac{1}{2}\{g_C, r\} + \frac{D_C-1}{2}\frac{D_C-3}{2}\frac{1}{r^2}\right] G_{l_C}(r, r_0, E_C, V_C(r), C_C) = \frac{\delta(r-r_0)}{r_0^{D_C-1}}\,. \quad (28)$$

The final form of the GF equation of the C-quantum system (C-QS) establish in Euclidean space of the chosen dimension D_C *is*

$$\left[\partial_r^2 + \frac{D_C-1}{r}\partial_r + E_C - V_C(r)\right] G_{l_C}(r, r_0, E_C, V_C(r), C_C) = \frac{\delta(r-r_0)}{r_0^{D_C-1}}\,, \quad (29)$$

where C-QS potential

$$V_C(r) = V_C^{(1)}(r) + V_C^{(2)}(r) + V_C^{(3)}(r). \quad (30)$$

Next we select $\frac{\lambda g_C^2}{1+\sigma g_C^2}$ as WP from equation (1) and using the ansatz (9) we obtained the transformation function

$$g_C(r) = \pm\eta^{1/2}\sqrt{\xi_n r^2 + 2r}\;. \quad (31)$$

The integration constant $C = \frac{1}{\sigma}$ implies that $g_C(r)$ has the desired local property of $g_C(0) = 0$, which will attribute asymptotic property $g_C(\infty) = \infty$ and we are consider the positive sign. In equation (31) $\eta = \sqrt{\frac{-E_C}{\lambda}}$ and $\sigma\eta = \xi_n$.

Leads to equation (10) with equation (31) yields

$$V_C^{(1)}(r) = C_C^2\frac{(\xi_n r + 1)^2}{(\xi_n r^2 + 2r)}\,, \quad (32)$$

C_C is the characteristic constant, where $C_C^2 = \eta(-E_B)$, further which will give the energy eigenvalues of c-exactly solvable quantum system (C-QS).

Using equation (11,31) we obtain

$$V_C^{(2)}(r) = \eta^2(\xi_n r + 1)^2, \quad (33)$$

and

$$V_C^{(3)}(r) = s(s+1)\frac{g_C'^2}{g_C^2} - \frac{1}{2}\{g_C, r\} - \frac{D_C-1}{2}\frac{D_C-3}{2}\frac{1}{r^2} \quad (D_A = 3), \quad (34)$$

where $l_A = s$ (for s wave). The multi-term potential of C-QS given by equation (30) becomes:

$$V_C(r) = C_C^2\frac{(\xi_n r+1)^2}{(\xi_n r^2 + 2r)} + \eta^2(\xi_n r + 1)^2 - \left\{s(s+1) - \frac{3}{4}\right\}\frac{(\xi_n r + 1)^2}{(\xi_n r^2 + 2r)^2} + \frac{3}{4}\frac{\xi_n^2}{(\xi_n r + 1)^2} - \frac{D_C-1}{2}\frac{D_C-3}{2}\frac{1}{r^2}. \quad (35)$$

The energy eigenvalues of $V_C(r)$ is

$$E_C = -\frac{\eta C_C^4}{\frac{\lambda}{\sigma} + 2s + 7}\,. \quad (36)$$

The presence of η and ξ_n in $V_C(r)$ make it Sturmian potential. To make it a normal/physical potential we have to make η and ξ_n, n independent which will make λ, n -dependent (equation (1)). Fortunately the combination $\sigma\eta$will be n-independent. Consequently the potential $V_C(r)$ becomes a normal potential by $\xi_n \to \xi$. In normal QS

$$V_C(r) = C_C^2 \frac{(\xi r+1)^2}{(\xi r^2+2r)} + \eta^2(\xi r+1)^2 - \left\{s(s+1) - \frac{3}{4}\right\}\frac{(\xi r+1)^2}{(\xi r^2+2r)^2} + \frac{3}{4}\frac{\xi_n^2}{(\xi r+1)^2} - \frac{D_C-1}{2}\frac{D_C-3}{2}\frac{1}{r^2}, \quad (37)$$

C-QS potential is identified as Shifted harmonic Oscillator. Energy eigenvalues of normal $V_C(r)$ is

$$E_C = \frac{\eta^2}{2}\left\{\xi(2s+7) + \sqrt{\xi^2(2s+7) + 4\xi C_c^4}\right\}. \quad (38)$$

Utilizing equation (3) and (6) we obtained the eigenfunction expansion of C-QS as:

$$G_C(r, r_0, E_C, V_C(r), C_C) = \sum_{n=0}^{\infty} \frac{\varphi_n^{(C)}(r)\varphi_n^{*(C)}(r)}{E - \frac{\eta^2}{2}\left\{\xi(2s+7) + \sqrt{\xi^2(2s+7)+4\xi C_c^4}\right\} - i\epsilon}. \quad (39)$$

From equation (39) the radial wave function is read off, which is

$$\varphi_n^{(C)}(r) = N_C r^{-\left(\frac{D_C-1}{2}\right)} \frac{(\xi_n r^2+2r)^{\frac{1}{2}\left(s+\frac{3}{2}\right)}}{(\xi_n r+1)^{\frac{1}{2}}} \left\{1 - \frac{\left(\frac{\xi}{E_C} - 4\eta\right)(\xi_n r^2+2r)}{2(2s+3)}\right\} exp\left\{-\frac{1}{2}\eta^2(\xi_n r^2+2r)^2\right\}. \quad (40)$$

The 2-D view of the potential, energy eigenvalues and eigenfunctions of C-QS are given in Fig. 2.

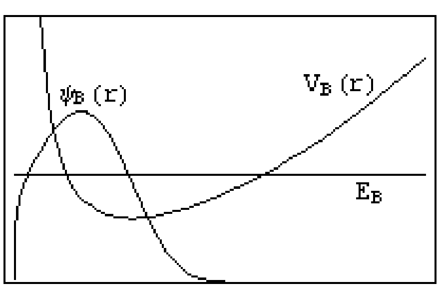

FIGURE 2. The continuous curves are for $V_C(r) = \frac{(4r+1)^2}{4r^2+2r} + (4r+1)^2 - \frac{5}{4}\frac{(4r+1)^2}{(4r^2+2r)^2} + \frac{12}{(4r+1)^2}$, $\varphi_n^{(C)}(r) = \frac{1}{4}\frac{(4r+1)^{5/4}}{(4r+1)^{1/2}}\{1 - 0.38(4r^2+2r)\}exp\left\{-\frac{1}{2}(4r^2+2r)\right\}$ and $E_C = 24.3$. The parameter sets are $C_C = 1, \xi = 4, \eta = 1, s = 0$.

IV. DISCUSSION AND CONCLUSION

In the present paper we have generated a class of exactly solved quantum systems in non relativistic quantum mechanics using Extended transformation method in any arbitrary dimensional Euclidean spaces within the framework of GF technique. Here the non-polynomial QS (A-system) potential is combination of power law and non-power law. Application of ET to the potential equation (1) choosing r^2 (power- law part) as working potential leads to the B-QS potential $V_B(r) = \frac{P}{r} + \frac{Q}{1+2Rr,}$ it is special interested in several areas of atomic and molecular physics. The A-QS potential with power-law, always generates the QS with a new power law potential. Successive application of ET on the generated QS will revert it back to the parent QS.

On the other hand the WP $\frac{\lambda r^2}{1+\sigma r^2}$ is a representative of non-power law potential and it leads to the C-QS potential $V_C(r) = C_C^2\frac{(\xi r+1)^2}{(\xi r^2+2r)} + \eta^2(\xi r+1)^2 - \left\{s(s+1) - \frac{3}{4}\right\}\frac{(\xi r+1)^2}{(\xi r^2+2r)^2} + \frac{3}{4}\frac{\xi_n^2}{(\xi r+1)^2} - \frac{D_C-1}{2}\frac{D_C-3}{2}\frac{1}{r^2}$. In non-power law cases the transformation function $g_C(r)$ is non-factorizable. This furnishes the special type of energy dependent non-power law potential (sturmian). This energy-dependent potential, unlike the usual QSs are always equipped with only a single normalized state, as varying n we do not get exited states, instead we get different QSs. Thus n is no longer a quantum number. It plays the role of a cardinal number parameter to enumerate different QSs. We discuss a method to regroup this set of energy-dependent QS to normal QS. The normal C-QS is qualitatively similar as Shifted harmonic Oscillator and we have not get any scattering states.

In conclusion we should point out that finding the GF for other potentials classes are also possible. The Coulomb, Fractional power potential and decemvirate power potential doubly enharmonic sextic power potentials are among such classes.

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