

A simple short derivation of the Clebsch-Gordan coefficients



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Abstract

We give a simple short derivation of the Clebsch-Gordan coefficients, motivated by simple introductory facts of quantum angular momentum. The derivation uses only binomial coefficients, eschewing raising- and lowering-operators, group theory, spinors and calculus.

Keywords: Quantum angular momentum coupling, Clebsch-Gordan coefficients.

Resumen

Damos una derivación sencilla de los coeficientes de Clebsch-Gordan, motivada por temas introductorios de momento angular cuántico. La derivación sólo utiliza coeficientes binomiales, evitando los operadores para subir o bajar índices, teoría de grupos, espinores y cálculo.

Palabras clave: Acoplamiento de momento angular cuántico, coeficientes de Clebsch-Gordan.

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I. INTRODUCTION

All courses on quantum mechanics include angular momentum, which is essential for understanding (say) atoms and molecules: how protons and electrons combine as atoms, and how atoms combine as molecules. In an atom the angular momentum of its many electrons add (combine, couple) to give the total angular momentum j of the atom.

Thus we wish to know how j_1 and j_2 add to give j : $j_1 + j_2 = j$. The answer is given by the Clebsch-Gordan coefficients (CGCs) $(j_1 j_2 m_1 m_2 | jm)$ which relate the angular momentum state $| jm \rangle$ to states $| j_1 m_1 \rangle$ and $| j_2 m_2 \rangle$:

$$| jm \rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \delta(m_1 + m_2, m) (j_1 j_2 m_1 m_2 | jm) | j_1 m_1 \rangle | j_2 m_2 \rangle. \quad (1)$$

We shall derive the CGCs, assuming the following elementary facts of $| jm \rangle$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$,

$m = -j, \dots, +j$, see §3:

- Any $| jm \rangle$ can be written in terms of $j = \frac{1}{2}$ as

$$| jm \rangle = \frac{\alpha^{j+m} \beta^{j-m}}{\sqrt{(j+m)!(j-m)!}}, \quad (2)$$

where $\alpha = | + \rangle \sim | j = \frac{1}{2}, m = +\frac{1}{2} \rangle$ and $\beta = | - \rangle \sim | j = \frac{1}{2}, m = -\frac{1}{2} \rangle$; this is also the intrinsic angular momentum (spin $s = \frac{1}{2}$) of an electron.

- For fixed j_1 and j_2 we have $j_1 + j_2 \leq j \leq |j_1 - j_2|$ (triangle rule), so the states $| 00 \rangle | jm \rangle$ and $| jm \rangle$ have the same j and m .

The state $\gamma = \alpha_1 \beta_2 - \alpha_2 \beta_1$ is $| 00 \rangle$, an invariant; $j = 0$ shows the zero-angular-momentum and no-preferred-direction of spherical symmetry. Thus γ^q is also an invariant: $| jm \rangle$ and $\gamma^q | jm \rangle$ have the same j and m . We say that they transform (\sim) in the same way (under rotation), $| jm \rangle \sim \gamma^q | jm \rangle$.

- We assume $\alpha \sim \alpha_1 + \alpha_2$ and $\beta \sim \beta_1 + \beta_2$ and note that $j = j_1 + j_2$.

Very few text-books on quantum mechanics derive CGCs: some quote the result (13) without proof, and most go no further (if that) than quoting (1). The very few derivations often use the 'advanced' unfamiliar mathematics of group theory and spinors. Hence we offer the following simple short derivation, *c.f.* [1].

II. DERIVATION

For fixed j_1 and j_2 we will derive the CGCs of the allowed j . Then §1 implies that

$$f(j, m) = [(\alpha_1 \beta_2 - \alpha_2 \beta_1)^q] \left[\frac{(\alpha_1 + \alpha_2)^{j+m} (\beta_1 + \beta_2)^{j-m}}{\sqrt{(j+m)!(j-m)!}} \right] \sim |jm\rangle. \quad (3)$$

Expanding (3) by the binomial theorem gives

$$f(j, m) = \sum_{r,t} (-1)^r \binom{q}{r} \binom{j+m}{p} \binom{j-m}{t} [\alpha_1^{p+r} \beta_1^{q-r+t} \alpha_2^{q-r-p+j+m} \beta_2^{r-t+j-m}] \quad (4)$$

where $\binom{a}{b} = a! / b!(a-b)!$ is a binomial coefficient.

Comparison of the $[\dots]_1$ and $[\dots]_2$ of (4) to (2) gives the substitutions

$$\begin{aligned} j_1 + m_1 &= p + r, & j_1 - m_1 &= q - r + t, \\ j_2 + m_2 &= q - r - p + j + m, & j_2 - m_2 &= r - t + j - m, \end{aligned} \quad (5)$$

which we manipulate to give $\sum_{m_1 m_2 r}$ by

$$p = j_1 + m_1 - r, \quad t = r - m + j - j_2 + m_2, \quad q = j_1 + j_2 - j. \quad (6)$$

Substituting (6) and (2) into (4) and simplifying (expand binomial coefficients and cancel factorials) then gives

$$\begin{aligned} f(j, m) &= \sum_{m_1 m_2} |j_1 m_1\rangle |j_2 m_2\rangle (-1)^{j_1 + j_2 - j} \delta(m_1 + m_2, m) \\ &\sum_r \frac{(-1)^r (j_1 + j_2 - j)! \sqrt{(j+m)!(j-m)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}{r!(j_1+j_2-j-r)!(j_1+m_1-r)!(j-j_1+m_2+r)!(j-m_1-j_2+r)!(j_2-m_2-r)!} \end{aligned} \quad (7)$$

Comparing (1) with (7) shows that the CGCs are effectively the \sum_r but we must ensure that $f(j, m)$ is normalised before we can call it $|jm\rangle$; this means that, with $m_2 = m - m_1$ we must have

$$N^2 \sum_{m_1} (j_1 j_2 m_1 m - m_1 | jm)^2 = 1, \quad (8)$$

from which we calculate the normalisation constant N . Since N is independent of m , we choose the special case $m = j$ and thus have $(j_1 j_2 m_1 j - m_1 | jj)$ which, from (7), has the single term $r = j_2 - m_2$ in \sum_r ; simplifying then gives

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$$\begin{aligned} (j_1 j_2 m_1 j - m_1 | jj) &= N (-1)^{j_2 - j + m_1} \\ &\sqrt{\frac{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m - m_1)!(j_2 - m + m_1)!}{(2j)!}} \\ &\begin{pmatrix} j_1 + j_2 - j \\ j_2 - j + m_1 \end{pmatrix} \begin{pmatrix} 2j \\ j_1 - j_2 + j \end{pmatrix}, \end{aligned} \quad (9)$$

Substituting (9) into (8) and simplifying gives

$$N^2 \begin{pmatrix} 2j \\ j_1 - j_2 + j \end{pmatrix}^2 \frac{(j_1 + j_2 - j)!^2}{(2j)!} \sum_{m_1} \frac{(j_1 + m_1)!(j_2 + j - m_1)!}{(j_2 - j + m_1)!(j_1 - m_1)!} = 1. \quad (10)$$

We now state that direct use of the properties of the binomial coefficients (no physics here) gives the \sum_{m_1} as [2],

$$\sum_{m_1} \frac{(j_1 + m_1)!(j_2 + j - m_1)!}{(j_2 - j + m_1)!(j_1 - m_1)!} = (j_2 - j_1 + j)!(j_1 - j_2 + j)! \begin{pmatrix} j_1 + j_2 + j + 1 \\ j_1 + j_2 - j \end{pmatrix}, \quad (11)$$

so, with (11) in (10),

$$N = \sqrt{\frac{(2j+1)(j_1 - j_2 + j)!(j_2 - j_1 + j)!}{(j_2 + j_1 - j)!(j_1 + j_2 + j + 1)!}}. \quad (12)$$

We now insert N of (12) into (8) and (7) and simplify, finally giving [2]

$$\begin{aligned} (j_1 j_2 m_1 m | jm) &= \delta(m_1 + m_2, m) \sqrt{\frac{(2j+1)(j_1 - j_2 + j)!(j_2 - j_1 + j)!(j_1 + j_2 - j)!}{(j_1 + j_2 + j + 1)!}} \\ &\sum_r \frac{(-1)^{r+j_2-j} \sqrt{(j+m)!(j-m)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}{r!(j_1+j_2-j-r)!(j_1+m_1-r)!(j-j_1+m_2+r)!(j-m_1-j_2+r)!(j_2-m_2-r)!} \end{aligned} \quad (13)$$

III. CONCLUDING REMARKS

We believe (hope) that our simple short derivation of the Clebsch-Gordan coefficients will be accessible to someone familiar only with some introductory facts of quantum angular momentum. The derivation uses only binomial coefficients: it avoids raising and lowering-operators, group theory, spinors and calculus. It also automatically gives $q = j_1 + j_2 - j$ in (3), and uses the simplest invariant $\gamma = (\alpha_1 \beta_2 - \alpha_2 \beta_1)$ -the familiar *singlet* state of a two-electron system.

Define the following operators [3, 2]:

$$j_+ = \alpha \frac{\partial}{\partial \beta}, \quad j_- = \beta \frac{\partial}{\partial \alpha}, \quad j_z = \alpha \frac{\partial}{\partial \alpha} - \beta \frac{\partial}{\partial \beta},$$

$$j_{\pm} = j_x \pm ij_y, \quad (14)$$

and apply $\mathbf{j} = \mathbf{j}(j_+, j_-, j_z)$ to (2) by simple differentiation:

$$j_{\pm} |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |jm \pm 1\rangle$$

$$j_z |jm\rangle = m |jm\rangle$$

$$\mathbf{j}^2 |jm\rangle = j(j+1) |jm\rangle. \quad (15)$$

Equations (2, 14, 15) represent the standard ‘abstract’ angular momentum states $|jm\rangle$, and show how to manipulate them via differentiation. In particular, for $j = 1/2$ and $\gamma = \alpha_1\beta_2 - \alpha_2\beta_1$, $\mathbf{j}^2\gamma$ is $[(j_x + j_{2x})^2 + (j_y + j_{2y})^2 + (j_z + j_{2z})^2][\alpha_1\beta_2 - \alpha_2\beta_1] = 0$; so $\gamma \sim |00\rangle$ and γ is an invariant, see §1. Again for $j = 1/2$, $j_-\alpha = \beta$ etc., so $\alpha \sim |1/2, 1/2\rangle$ etc.; similarly $(j_- + j_{2-})(\alpha_1 + \alpha_2) = (\beta_1\partial/\partial\alpha_1 + \beta_2\partial/\partial\alpha_2)(\alpha_1 + \alpha_2) = (\beta_1 + \beta_2)$, so $(\alpha_1 + \alpha_2) \sim |1/2, 1/2\rangle$, and thus $\alpha_1 + \alpha_2 \sim \alpha$ etc., as required in §1.

We have implied that (3) is ‘reasonable’. Let $n = 1 \dots 3$, $j = j_3$, $x = (\mathbf{j}, \alpha, \beta)$, $x_3 = x_1 + x_2$, $n^* = (j_n + m_n)!(j_n - m_n)!$ and let f_3 and f_{12} mean f evaluated in states $|j_3 m_3\rangle$ and states $|j_1 m_1\rangle |j_2 m_2\rangle$. Then our logic flow is: $|jm\rangle \equiv |jm\rangle$, $|00\rangle |jm\rangle \equiv |00\rangle |jm\rangle$, $[|00\rangle |jm\rangle]_3 \equiv [|00\rangle |jm\rangle]_{12}$, $|00\rangle_3 |jm\rangle_3 = N |00\rangle_{12} |jm\rangle_{12}$, $|00\rangle_3 = 1$, $|00\rangle_{12} = (\alpha_1\beta_2 - \alpha_2\beta_1)^q$, $|jm\rangle_3 = |j_3 m_3\rangle = \alpha_3^{j_3+m_3} \beta_3^{j_3-m_3} / \sqrt{3^*}$, $|jm\rangle_{12} = |j_3 m_3\rangle_{12} = (\alpha_1 + \alpha_2)^{j_3+m_3} (\beta_1 + \beta_2)^{j_3-m_3} / \sqrt{3^*}$, $\alpha_3^{j_3+m_3} \beta_3^{j_3-m_3} = N(\alpha_1\beta_2 - \alpha_2\beta_1)^q (\alpha_1 + \alpha_2)^{j_3+m_3} (\beta_1 + \beta_2)^{j_3-m_3}$.

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