Tailoring the law of kinetic Energy to the initial instant problem

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Abstract

Neat yet colorful examples are indispensable to classroom lecturing and examination, however most dynamical examples in existing textbooks are of wheel system with pointed masses. Here we discuss the systems containing planar motion bars, but limited to the initial instant problem (IIP). By using the IIP's peculiarity, the general explicit expression for the kinetic energy can be circumvented, which can shorten the analysis procedure significantly. Similar ideas can be extended to the system of two degrees of freedom, with supplemented an extra equation.

Keywords: Pendulum, Vibration, Multi-degree-of-freedom.

Resumen

Ejemplos atractivos y coloridos son indispensables para las clases y exámenes en el aula, sin embargo la mayoría de los ejemplos de dinámica en los libros de texto existentes son sistemas de ruedas con masas puntuales. Aquí discutimos sistemas que contienen barras de movimiento plano, pero limitados al problema inicial instantáneo (PII). Mediante el uso de la peculiaridad de PII, la expresión explícita general para la energía cinética se puede eludir, lo cual puede reducir el procedimiento del análisis de manera significativa. Ideas similares se pueden extender para el sistema de dos grados de libertad, complementado con una ecuación adicional.

Palabras clave: Péndulo, vibración, varios grados de libertad.

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I. INTRODUCTION

The dynamics teaching and its examination necessitate concise yet colorful examples for the sake of the limitation of class hours and blackboard or slide spaces. Although dynamical problems exist ubiquitously in engineering, examples with easiness appropriate for class lecture and examination are not so many. Examples in most textbooks use wheel systems with pointed masses, e.g. the system of FIGURE 1(a), although bars are used extensively in engineering applications. We definitely desire for examples containing bars to enhance teaching and examination diversity, since the range of variety of wheels and masses is limited and monotonic.

The case shown in FIGURE 1(b) contains one bar. Even though there is only one bar, the general solution is too tedious to be lectured on a blackboard. As a result of tradeoff between variety and conciseness, some so-called initial instant examples as in FIGURE 1 (b~d) are recommended [1]. The "initial instant problem(IIP)" refers to the dynamic analysis at the very beginning when a system KEPT at REST is RELEASED, for example, releasing the bar *AB* in FIGURE 1(b), or severing the light rope *OA* in FIGURE 1(d).

The simplicity pertaining to IIPs is that the velocities, both the linear and angular, are zero, which can be used to

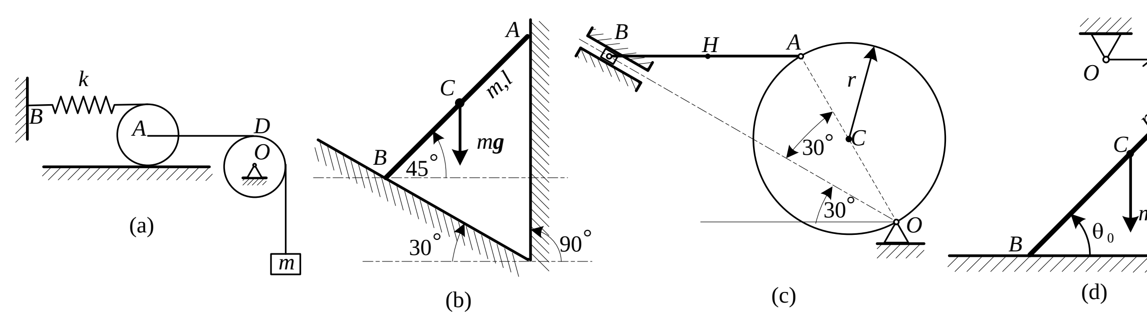


FIGURE 1. Classical examples diagram of dynamical problems.

shorten analysis procedure. However, because the general explicit expression for kinetic energy is usually not easy to write out (for example FIGURE 1 (b~d)), the IIPs are often analyzed with the ordinary differential equation (ODE) based approach [2]. This approach requires an acceleration analysis beforehand (see the next section), which is cumbersome for students.

Usually the Law of Kinetic Energy (LKE) is more efficient than the approach based on ODEs because the former can avoid an acceleration analysis, but the cost is to write out the general explicit expression for the kinetic energy. We will demonstrate that this is not necessary with IIPs after their peculiarity being taken into account.

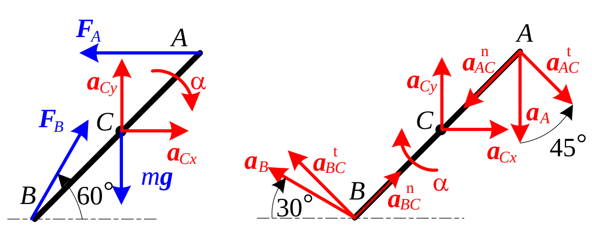
In the following, we first experience the sophistication of the approach based on ODEs, then we illustrate how the IIP's peculiarity can be used to save the LKE based approach from exact explicit expressions of kinetics. Finally, this will be extended to cases of two degrees of freedom.

II. CONVENTIONAL APPROACH

For problems involving planar motion bars, conventionally, we use ODEs for a planar motion to solve them, rather than the LKE based approach [3]. This is because the LKE based approach often needs a general explicit expression of kinetic energy, which is rather challenging for systems containing planar motion bars. Although velocities in IIPs are zero at the very beginning, this peculiar fact has not caught much attention, because the kinetic energy is zero just at the initial instant.

Overall the differential equation based approach is rather sophisticated. We will illustrate this entanglement through the example of FIGURE 1(b) in this section.

In this example we assume that the wall is frictionless and the bar *AB* is uniform with a mass *m* and a length *l*. The question is how much is the angular acceleration of *AB* when it is released from the position shown in the figure.



(a) (b)

FIGURE 2. Diagram of forces on a bar.

First we draw the free body diagram of forces on the bar as shown in FIGURE 2(a), where $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ are the normal forces exerted on points *A* and *B*, respectively. In the figure $\boldsymbol{a}_{Cx}$ and $\boldsymbol{a}_{Cy}$ are components of the acceleration of the mass center *C*. The angular acceleration is denoted as the symbol α. The ordinary differential equations for the *AB* are

$$\left.\begin{aligned} ma_{Cx} &= \sum F_x = -F_A + F_B\cos 60^\circ \\ ma_{Cy} &= \sum F_x = -mg + F_B\sin 60^\circ \\ \frac{1}{12}ml^2\alpha &= \sum M_C(F) = -F_A\frac{l}{2}\cos 45^\circ + F_B\frac{l}{2}\sin 15^\circ \end{aligned}\right\} \quad (1)$$

Equation (1) involves five unknowns, but there are only three equations. We need two more equations from the acceleration analysis, which is shown in FIGURE 2(b). In the figure the relative accelerations of points *A* and *B* with respect to *C* along the normal directions, $\boldsymbol{a}_{AC}^{\mathrm{n}}$ and $\boldsymbol{a}_{BC}^{\mathrm{n}}$, are depicted deliberately. Both are zero due to the zero angular velocity of *AB*, the IIP's peculiarity. The wall constraint renders the accelerations of *A* and *B* to be along the wall.

In light of FIGURE 2(b), we have

$$\left.\begin{aligned} \boldsymbol{a}_A &= \boldsymbol{a}_C + \boldsymbol{a}_{AC}^{\mathrm{t}} = \boldsymbol{a}_{Cx} + \boldsymbol{a}_{Cy} + \boldsymbol{a}_{AC}^{\mathrm{t}} \\ \boldsymbol{a}_B &= \boldsymbol{a}_C + \boldsymbol{a}_{BC}^{\mathrm{t}} = \boldsymbol{a}_{Cx} + \boldsymbol{a}_{Cy} + \boldsymbol{a}_{BC}^{\mathrm{t}} \end{aligned}\right\} \quad (2)$$

We project the two equations of Equation (2) to the directions perpendicular to $\boldsymbol{a}_A$ and $\boldsymbol{a}_B$ correspondingly, and obtain

$$\left.\begin{aligned} 0 &= a_{Cx} + a_{AC}^{\mathrm{t}}\cos 45^\circ \\ 0 &= a_{Cy}\cos 30^\circ + a_{Cx}\sin 30^\circ + a_{BC}^{\mathrm{t}}\sin 15^\circ \end{aligned}\right\} \quad (3)$$

Substituting $a_{AC}^{\mathrm{t}} = a_{BC}^{\mathrm{t}} = l\alpha/2$ into equation (3) leads to

$$\left.\begin{aligned} a_{Cx} &= -\sqrt{2}l\alpha/4 \\ a_{Cy} &= (2\sqrt{6} - 3\sqrt{2})l\alpha/12 \end{aligned}\right\} \quad (4)$$

Then substituting equation (4) back into equation (1) and eliminating F_A and F_B, we eventually arrive at

$$\alpha = \frac{\sqrt{2} - \sqrt{6}}{4}\frac{g}{l}. \quad (5)$$

III. TAILORED LKE BASED APPROACH

It should be pointed that there is ONLY ONE bar in the problem of FIGURE 1(b). For more sophisticated systems, e.g. FIGURE 1(c), the difficulty with the conventional approach is intolerable. In contrast, the LKE is apt to the single degree-of-freedom (SDOF) system with multiple members. This law states as follows

$$T_2 - T_1 = \Delta W, \quad (6)$$

where T_1 and T_2 are the kinetic energy at two instants, and ΔW is the work increment done by the active forces between the two instants.

As for the IIP we discussed here, there is only one instant—the initial instant; accordingly, we must use the power formulas—the derivative form of equation (6), that is

$$\frac{\mathrm{d}T}{\mathrm{d}t}=\sum P_i=\sum \boldsymbol{F}_i\cdot\boldsymbol{v}_i=\sum F_i^{\mathrm{t}}v_i\ , \qquad (7)$$

where P_i is the power of the individual force (or moment) $\boldsymbol{F}_i$. The power of force $\boldsymbol{F}_i$ equals to the vectorial dot product of the force vector $\boldsymbol{F}_i$ and the velocity vector $\boldsymbol{v}_i$ of the point exerted by $\boldsymbol{F}_i$. It can also be expressed as a scalar product of v_i and $\boldsymbol{F}_i^{\mathrm{t}}$, the tangential component of $\boldsymbol{F}_i$ along the direction of $\boldsymbol{v}_i$.

The bar AB of FIGURE 1(b) at an arbitrary position is illustrated in FIGURE 3(a). The instantaneous velocity

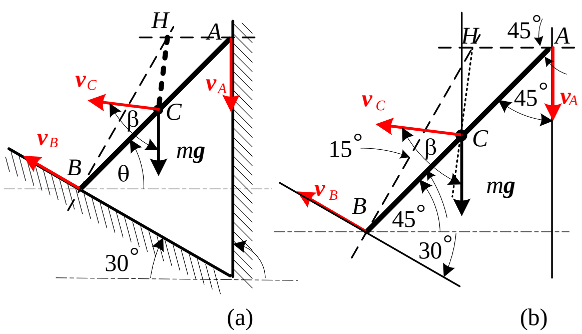


FIGURE 3. Diagram of forces on a bar at an arbitrary position.

center H can be determined easily by drawing two lines perpendicular to the velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$, and the intersection of the two lines is just the instantaneous center. Accordingly the kinetic energy is

$$T=\frac{1}{2}J_H\dot{\theta}^2=\frac{1}{2}\left(\frac{ml^2}{12}+mHC^2\right)\dot{\theta}^2\,. \qquad (8)$$

For a general case, we need an explicit expression of HC with respect to the variable θ since it is involved in derivative operations of equation (7), but now we just keep it there. The only active force is the gravity of AB and its power is

$$P=v_C mg\cos\beta=HC\dot{\theta}\cos\beta\ , \qquad (9)$$

where β stands for the angle between the velocity $\boldsymbol{v}_C$ and gravity $m\boldsymbol{g}$. The explicit expression of β is very complex also. However since it is not involved in the derivative operations, we just need the particular value at the initial instant.

Subsituting equations (8) and (9) back into (7), we obtain

$$\left(\frac{ml^2}{12}+mHC^2\right)\dot{\theta}\ddot{\theta}+mHC\frac{\mathrm{d}HC}{\mathrm{d}t}\dot{\theta}^2=HC\dot{\theta}mg\cos\beta\,.$$

Dividing both sides by $\dot{\theta}$ leads to

$$\left(\frac{ml^2}{12}+mHC^2\right)\ddot{\theta}+mHC\frac{\mathrm{d}HC}{\mathrm{d}t}\dot{\theta}=HCmg\cos\beta\,. \qquad (10)$$

Now let us look into the peculiarity at the initial instant. The angular velocity $\dot{\theta}$ is zero. Besides, HC and its change rate $\frac{\mathrm{d}HC}{\mathrm{d}t}$ are both bounded, as a result, the second term on the left-hand side of equation (10) is zero. Hence at the initial instant we have

$$\left(\frac{ml^2}{12}+mHC^2\right)\ddot{\theta}=HCmg\cos\beta\,. \qquad (11)$$

The remaining task is to determine HC and β at the initial instant, rather than their general explicit expressions.

In FIGURE 3(b), the two sides of the angle β are perpendicular to the corresponding two sides of $\angle CHA$, thus $\beta=\angle CHA$. Applying the cosine theorem to ΔAHC leads to

$$HC\cos\beta=\frac{HA^2+HC^2-AC^2}{2HA}\,, \qquad (12)$$

where HA can be determined from ΔAHB using the sine theorem as

$$\frac{HA}{\sin\angle ABH}=\frac{AB}{\sin\angle AHB}\,. \qquad (13)$$

That is

$$HA=l\frac{\sin 15^\circ}{\sin 120^\circ}=\frac{3\sqrt{2}-\sqrt{6}}{6}l\ , \qquad (14)$$

HC^2 in equation (12) and (11) can be determined from ΔAHC using the cosine theorem as

$$HC^2=HA^2+AC^2-2HA\cdot AC\cos\angle HAC\,. \qquad (15)$$

With equation (14), $AC=l/2, \angle HAC=45^\circ$, HC^2 is

$$HC^2=\frac{(5-2\sqrt{3})}{12}l^2\,. \qquad (16)$$

The same result as equation (5) can be obtained by substituting equations (12), (14) and (16) into equation (11).

Since the general expression for the kinetic energy and its derivative are not involved, this tailored LKE based approach is much simpler compared to the conventional approach in the last section.

IV. GENERALIZATION

The tailored LKE based approach in the previous section can be generalized to any steady SDOF system.

Assume s is the generalized coordinate describing the SDOF system. Then the displacement vector $\boldsymbol{r}_i$ of any point is a function of s, thus

$$\boldsymbol{v}_i = \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}s}\dot{s} \,. \qquad (17)$$

The total kinetic energy is

$$T = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i = \frac{1}{2}\tilde{m}(s)\dot{s}^2 \,, \qquad (18)$$

where $\tilde{m}(s) = \sum m_i \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}s}\cdot\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}s}$ is the generalized mass.

The power in equation (7) equals to

$$P = \sum \boldsymbol{F}_i \cdot \boldsymbol{v}_i = \sum \boldsymbol{F}_i \cdot \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}s}\dot{s} = Q(s)\dot{s} \,, \qquad (19)$$

where $Q(s)$ is the generalized force corresponding to s.

Substituting equations (18) and (19) into equation (17) leads to

$$\frac{1}{2}\frac{\mathrm{d}\tilde{m}(s)}{\mathrm{d}s}\dot{s}^3 + m(s)\dot{s}\ddot{s} = Q\dot{s} \,. \qquad (20)$$

It can be further reformulated as

$$\ddot{s} = \frac{1}{\tilde{m}(s)}\left[Q(s) - \frac{1}{2}m'(s)\dot{s}^2\right]. \qquad (21)$$

Consider the peculiarity of the IIP case, $\dot{s}=0$, the above equation is reduced to

$$\ddot{s} = Q(s)/\tilde{m}(s) \,. \qquad (22)$$

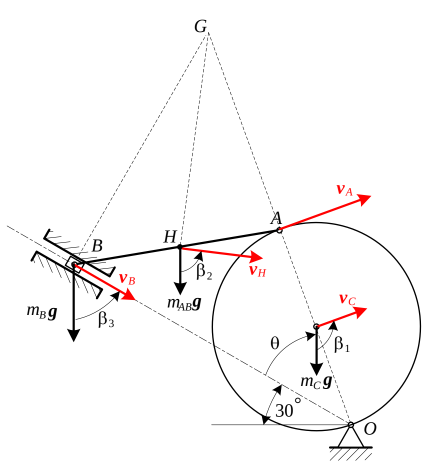


FIGURE 4. Diagram of forces on a bar and a disk.

This is parallel to the Newton second law of $a = F/m$.

Now let us appreciate the above generalization with a more sophisticated case of FIGURE (1c), and some tricks for analysis. In this example, the joints and slide are frictionless, and the bar AB and dick C are uniform with masses m_{AB}, m_C respectively. The point mass B has a mass m_B. $AB = AO = 2r$. AB is horizontal at the initial instant. We want to know the angular acceleration of the disc C when the system is released from the configuration in FIGURE 1(c).

FIGURE 4 shows the system configuration in an arbitrary configuration. The system kinetic energy is

$$\begin{aligned} T &= \frac{3}{4}m_C r^2\dot{\theta}_C^2 + \frac{1}{2}m_{AB}\left(\frac{AB^2}{12} + GH^2\right)\omega_{AB}^2 + \frac{1}{2}m_B GB^2\omega_{AB}^2 \\ &= \frac{3}{4}m_C r^2\dot{\theta}_C^2 + \frac{1}{2}\left(m_{AB}\frac{AB^2}{12} + m_{AB}GH^2 + m_B GB^2\right)\omega_{AB}^2 \end{aligned} \qquad (23)$$

Still we do not need figure out the explicit relationship between $\dot{\theta}_C$ and ω_{AB}. The force power is

$$\begin{aligned} P = {} & m_C gR\dot{\theta}_C \cos\beta_1 + m_{AB} gGH\omega_{AB}\cos\beta_2 + \\ & m_B g\,BH\omega_{AB}\cos\beta_3 \end{aligned} \,, \qquad (24)$$

where $\beta_1, \beta_2, \beta_3$ are indicated in FIGURE 4. In the figure G is the instantaneous velocity center of AB.

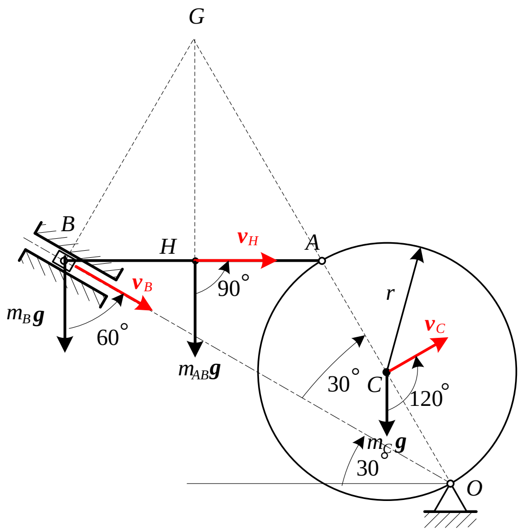
Substituting equations (23) and (24) into (7) and dividing both sides by $\dot{\theta}_C$, we obtain

$$\begin{aligned} & \frac{3}{2}m_C r^2\ddot{\theta}_C + \left(m_{AB}\frac{AB^2}{12} + m_{AB}GH^2 + m_B GB^2\right)\frac{\omega_{AB}}{\dot{\theta}_C}\alpha_{AB} \\ & + \left(m_{AB}GH\frac{\mathrm{d}GH}{\mathrm{d}t} + m_{AB}GB\frac{\mathrm{d}GB}{\mathrm{d}t} + m_B GB^2\right)\frac{\omega_{AB}}{\dot{\theta}_C}\omega_{AB} = \\ & m_C gR\cos\beta_1 + m_{AB} gGH\frac{\omega_{AB}}{\dot{\theta}_C}\cos\beta_2 + \\ & m_B g\,BH\frac{\omega_{AB}}{\dot{\theta}_C}\cos\beta_3 \end{aligned} \qquad (25)$$

Since at the initial instant, both $\dot{\theta}_C$ and ω_{AB} are zero, as a result, $\omega_{AB}/\dot{\theta}_C$ is the indeterminate 0/0. For a dynamical problem, this 0/0 should be understood as the limit of $\omega_{AB}(t)/\dot{\theta}_C(t)$ as time t approaches 0 from t>0. Concerning the SDOF system, the velocity ratio depends on the geometrical configuration only. Thus, we can determine the velocity ratio limit through a finite nonzero situation.

As shown in FIGURE 5 of the initial instant, it can be verified that $GH = GA = AO = 2r = GB = AB$. Considering the velocity of the common point A of the disk C and bar AB, we have

$$\frac{\omega_{AB}}{\dot{\theta}_C}=\frac{GA}{AO}=1. \qquad (26)$$



(b)

FIGURE 5. Diagram of forces on a bar and a disk.

For a realistic physical system, the quantity in the second round bracket in equation (25) is bounded, as a result,

$$\left(m_{AB}GH\frac{\mathrm{d}GH}{\mathrm{d}t}+m_{AB}GB\frac{\mathrm{d}GB}{\mathrm{d}t}+m_B GB^2\right)\frac{\omega_{AB}}{\dot{\theta}_C}\omega_{AB}=0. \qquad (27)$$

In light of the configuration in FIGURE 5, we have

$$\beta_1=120^\circ, \beta_2=90^\circ, \beta_3=60^\circ. \qquad (28)$$

Substituting equations (26), (27) and (28), we eventually arrive at

$$\ddot{\theta}_C=\frac{2m_B-m_C}{24m_B+20m_{AB}+9m_C}\frac{3g}{r}.$$

V. EXTENTIONS TO TWO-DOF CASES

Definitely we attempt to extend the applicable scope of the tailored LKE approach. However, this approach cannot be applied to systems of two degrees of freedom (DOF) directly. This is because either LKE or its derivative-power formulas equation, is a scalar relationship, and it can produce just ONE equation. For a two-DOF system, in general, we need two equations for individual generalized variables. An extra equation need be supplemented.

The system in FIGURE 1(d) is a little bit simpler, though it has two DOFs after the light rope severed. This is because no force is exerted along the horizontal direction, and the law of conservation of momentum along this direction can be employed. In brief, from mathematical view, this system has one free variable still. Further more, the horizontal component of the velocity of the mass center C is zero, thus the orbit of C is vertical down.

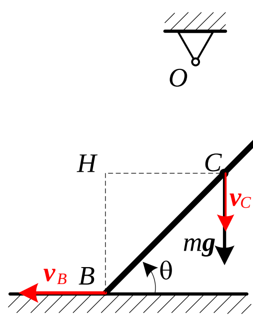


FIGURE 6. A two-DOF system.

The motion analysis has been depicted in FIGURE 6, where H is the instantaneous velocity center of AB. The system kinetic energy at an arbitrary position is

$$T=\frac{1}{2}J_H\dot{\theta}^2=\frac{1}{2}\left(\frac{ml^2}{12}+mHC^2\right)\dot{\theta}^2. \qquad (29)$$

And the corresponding power is

$$P=mgHC\dot{\theta}. \qquad (30)$$

Substituting equation (29) and (30) into (7) leads to

$$\left(\frac{ml^2}{12}+mHC^2\right)\ddot{\theta}+mHC\frac{dHC}{dt}\dot{\theta}=mgHC. \qquad (31)$$

Although in this case the relationship between HC and θ is very simple, we still keep it implicitly. For the IIP, the second term of equation (31) is discarded straightforwardly, and we have

$$\ddot{\theta}=gHC\left(\frac{l^2}{12}+HC^2\right)^{-1}. \qquad (33)$$

At the initial instant, $HC=l/2\cos\theta_0$, and equation (33) is reduced to the final solution as

$$\ddot{\theta}=6\frac{\cos\theta_0}{1+3\cos^2\theta_0}\frac{g}{l}.$$

VI. CONSTITUTIVE TWO-DOF CASES

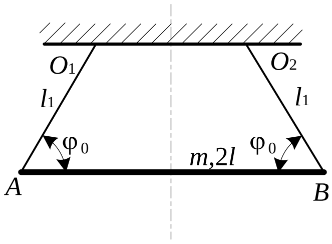


FIGURE 7. Case of two-DOF.

Now we consider a nontrivial two-DOF case shown in FIGURE 7, where the bar *AB* is uniform with a length 2*l* and mass *m*, and the soft light ropes O_1A and O_2B have the same length l_1, whereas their masses are ignored. We want to know the angular acceleration of *AB* at the initial instant after O_2B is severed. In this case, no conservation law can be used. We start from the kinetic energy at an arbitrary position in FIGURE 8 as the following

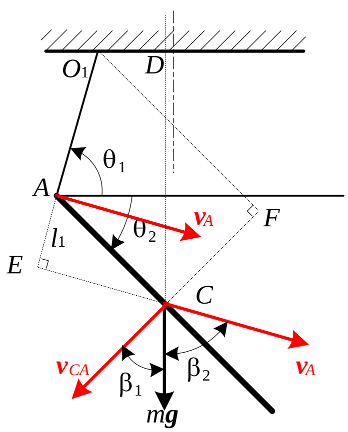


FIGURE 8. Case of two-DOF.

$$T=\frac{1}{2}\frac{m(2l)^2}{12}\dot\theta_2^2+\frac{1}{2}m[(l_1\dot\theta_1)^2+(l\dot\theta_2)^2+2l_1l\dot\theta_1\dot\theta_2\cos(\beta_1+\beta_2)], \tag{34}$$

where β_1,β_2 are illustrated in the FIGURE 8. We still keep them there without explicit expressions with respect to the generalized variables θ_1,θ_2.

The force power in the configuration of FIGURE 8 is

$$P=mgv_A\cos\beta_2+mgv_{BA}\cos\beta_1=mgl_1\dot\theta_1\cos\beta_2+mgl\dot\theta_2\cos\beta_1\ . \tag{35}$$

Substituting equations (34) and (25) into (7) verifies that

$$\frac{ml^2}{3}\dot\theta_2\ddot\theta_2+m[l_1^2\dot\theta_1\ddot\theta_1+l^2\dot\theta_2\ddot\theta_2+l_1l\ddot\theta_1\dot\theta_2\cos(\beta_1+\beta_2)+l_1l\dot\theta_1\ddot\theta_2\cos(\beta_1+\beta_2)-l_1l\dot\theta_1\dot\theta_2(\dot\beta_1+\dot\beta_2)\sin(\beta_1+\beta_2)]=mgl_1\dot\theta_1\cos\beta_2+mgl\dot\theta_2\cos\beta_1. \tag{36}$$

Dividing both sides with $\dot\theta_1$ leads to

$$\frac{ml^2}{3}\frac{\dot\theta_2}{\dot\theta_1}\ddot\theta_2+m[l_1^2\ddot\theta_1+l^2\frac{\dot\theta_2}{\dot\theta_1}\ddot\theta_2+l_1l\ddot\theta_1\frac{\dot\theta_2}{\dot\theta_1}\cos(\beta_1+\beta_2)+l_1l\ddot\theta_2\cos(\beta_1+\beta_2)-l_1l\dot\theta_2(\dot\beta_1+\dot\beta_2)\sin(\beta_1+\beta_2)]=mgl_1\cos\beta_2+mgl\frac{\dot\theta_2}{\dot\theta_1}\cos\beta_1. \tag{37}$$

According to the IIP's peculiarity, the fifth term in the square bracket is zero,

$$l_1l\dot\theta_2(\dot\beta_1+\dot\beta_2)\sin(\beta_1+\beta_2)=0\ . \tag{38}$$

Contrary to the SDOF case from the geometrical information, we cannot determine the ratio $\dot\theta_2/\dot\theta_1$ now. This system has two DOFs and θ_1,θ_2 are independent of each other. We understand the ratio $\dot\theta_2/\dot\theta_1$ is the limit of $\dot\theta_2(t)/\dot\theta_1(t)$ as t approaching zero. From mathematical view, we have

$$\frac{\dot\theta_2(t)}{\dot\theta_1(t)}=\frac{\dot\theta_2(0)+\int_0^t\ddot\theta_2(t)\mathrm{d}t}{\dot\theta_1(0)+\int_0^t\ddot\theta_1(t)\mathrm{d}t}=\frac{\int_0^t\ddot\theta_2(t)\mathrm{d}t}{\int_0^t\ddot\theta_1(t)\mathrm{d}t}. \tag{39}$$

According to L'Hopital rule, we have

$$\lim_{t\to0}\frac{\int_0^t\ddot\theta_2(t)\mathrm{d}t}{\int_0^t\ddot\theta_1(t)\mathrm{d}t}=\lim_{t\to0}\frac{\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\ddot\theta_2(t)\mathrm{d}t)}{\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\ddot\theta_1(t)\mathrm{d}t}=\lim_{t\to0}\frac{\ddot\theta_2(t)}{\ddot\theta_1(t)}=\frac{\ddot\theta_2(0)}{\ddot\theta_1(0)}\ .$$

As a result

$$\frac{\dot\theta_2}{\dot\theta_1}\overset{\Delta}{=}\lim_{t\to0}\frac{\dot\theta_2(t)}{\dot\theta_1(t)}=\frac{\ddot\theta_2(0)}{\ddot\theta_1(0)}\overset{\Delta}{=}\frac{\ddot\theta_2}{\ddot\theta_1}\ . \tag{40}$$

Equation (40) means that the velocity ratio limit equals to the ratio of finite accelerations in IIPs.

Further at the initial instant we can figure out (see FIGURE 8)

$$\beta_1=0,\beta_2=\varphi_0\ . \tag{41}$$

Substituting equations (38), (40) and (41) into equation (37), we have

$$4l^2\ddot{\theta}_2^2/3+l_1^2\ddot{\theta}_1^2+2l_1l\ddot{\theta}_1\ddot{\theta}_2\cos\varphi_0=gl_1\cos\varphi_0\ddot{\theta}_1+gl\ddot{\theta}_2. \quad (42)$$

As we expect, there are two unknowns in the above equation. We must supplement another equation. Here we use the theorem of angular momentum around a fixed point. We can write out the system angular momentum around the fixed point O_1 as,

$$L_{O_1}=-\frac{m(2l)^2}{12}\dot{\theta}_2+(ml_1\dot{\theta}_1)O_1E-(ml\dot{\theta}_2)O_1F. \quad (43)$$

The external force produces an external moment of force around the O_1 as

$$M_{O_1}(F)=-mgO_1D=-mg(l\cos\theta_2-l_1\cos\theta_1). \quad (44)$$

According to the angular momentum theorem

$$\frac{\mathrm{d}L_{O_1}}{\mathrm{d}t}=M_{O_1}(F).$$

We have

$$-\frac{m(2l)^2}{12}\ddot{\theta}_2+mO_1E\,l_1\ddot{\theta}_1-mO_1F(l\ddot{\theta}_2)+ ml_1\dot{\theta}_1\frac{\mathrm{d}O_1E}{\mathrm{d}t}-ml\dot{\theta}_2\frac{\mathrm{d}O_1F}{\mathrm{d}t}=-mg(l\cos\theta_2-l_1\cos\theta_1). \quad (45)$$

As we argue in using LKE for the IIP,

$$ml_1\dot{\theta}_1\frac{\mathrm{d}O_1E}{\mathrm{d}t}=ml\dot{\theta}_2\frac{\mathrm{d}O_1F}{\mathrm{d}t}=0. \quad (46)$$

At the Initial instant we have the following geometrical parameters

$$\begin{aligned}&\theta_1=\varphi_0,\theta_2=0\\&O_1D=l-l_1\cos\varphi_0\\&O_1E=l_1-l\cos\varphi_0\\&O_1F=O_1D=l-l_1\cos\varphi_0\end{aligned} \quad (47)$$

Substituting equations (46) and (47) into (45), we have

$$(4l/3-l_1\cos\varphi_0)l\ddot{\theta}_2-(l_1-l\cos\varphi_0)l_1\ddot{\theta}_1=g(l-l_1\cos\varphi_0). \quad (48)$$

Solving the simultaneous equations of (42) and (48), we obtain the final solution

$$\ddot{\theta}_2=\frac{3\sin^2\varphi_0}{1+3\sin^2\varphi_0}\frac{g}{l}. \quad (48)$$

Solution (48) has nothing to with l_1.

A common case is $\varphi_0=90^\circ,\ddot{\theta}_2=3g/(4l)$. For this case, O_1A and O_2B are parallel to the gravitational acceleration. This means that the horizontal external force component is zero at the initial instant. The extra equation can be supplemented by this fact, that is, the horizontal acceleration component of the mass center C is zero. This fact can be further rendered by the O_1A as $\ddot{\theta}_1=0$. With this equation, we can directly get the solution $\ddot{\theta}_2=3g/(4l)$ from equation (42).

|Other common cases are: $\varphi_0=45^\circ,\ddot{\theta}_2=3g/(5l)$; $\varphi_0=60^\circ,\ddot{\theta}_2=9g/(11l)$; $\varphi_0=30^\circ,\ddot{\theta}_2=3g/(7l)$.

In addition, solution (48) is applicable to the case of $\varphi_0>90^\circ$.

VII. CONCLUSION

We have illustrated a tailored LKE based approach to the initial instant problems (IIP). Using this approach, the IIP can be lectured concisely with limited time and blackboard space, since the general explicit expression for the kinetic energy can be avoided. As a result, the dynamical examples are no longer the monotonic wheel systems.

The conciseness of the tailored LKE approach is due to the facts: 1) the physical realistic system is bounded; 2) the ratio limit of velocities at the initial instant equals to the ratio of the corresponding finite accelerations.

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