Solution of Einstein's Field Equations for an accelerated magnetic wave

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Abstract

In this paper, some solutions of Einstein's field equations for stress-energy tensor and magnetic stress tensor expressed in classical terms are proposed, which are combined to form a "magnetic stress-energy tensor" for an accelerated magnetic wave.

Keywords: Exact solutions, Stress-energy tensor, Magnetic stress tensor.

Resumen

En este trabajo, son propuestas algunas soluciones de las ecuaciones de campo de Einstein para el tensor de tensión-energía y el tensor de tensión magnética en términos clásicos, los cuales son combinados para formar un "tensor magnético de tensión-energía" para una onda magnética acelerada.

Palabras clave: Soluciones exactas, Tensor de tensión-energía, Tensor de tensión magnética.

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I. INTRODUCTION

In General Relativity (GR), Einstein's field equations [1] describe the metric of spacetime as well as its dynamical behaviour. Solutions of Einstein's field equations are called "exact solutions" [2, 3] which are metrics of spacetime (hence often called "metrics") and describe the structure of the spacetime including the motion of objects (as particles and charges) in spacetime.

Current cosmological principle considers the universe as homogenous, isotropic and accelerating [4, 5] (like a spherical surface in accelerated dilation, for simplicity), which define a specific metrics of spacetime, where general relativity is applicable.

Solutions of Einstein's field equations usually include certain tensor fields which are taken to model states of ordinary matter, giving specific contributions to the stress-energy tensor [6], which is a tensor quantity that describes the density and flux of energy and moment in spacetime according to the continuity equations. This tensor includes the energy components capable to distort and curve spacetime accounting the presence of matter, electromagnetic fields, and other physical effects that contribute to the mass-energy at the spacetime, being the source of the gravitational field in the Einstein's field equations.

One of the most known tensor solutions is the Maxwell stress tensor [7] for the electromagnetic field, which comes from the Maxwell's equations [8]. In addition, some works describe the field evolution equations including magnetic field from tensor analysis [9], showing the contribution of the "magnetic stress-energy tensor" in the Einstein's field equations. Previous works attempt to unify both; electromagnetic stress-energy tensor and gravity [10, 11], which consider that magnetic field could contribute in gravity to form a specific geometry of spacetime. Furthermore, previous papers consider that electromagnetic wave can propagate in four-dimension spacetime [12], as well as its propagation in a uniformly accelerated simple medium [13].

In this paper, some solutions of Einstein's field equations for stress-energy tensor and magnetic stress tensor expressed in classical terms are proposed, which are combined to form a magnetic stress-energy tensor for an accelerated magnetic wave.

II. SOLUTION OF EINSTEIN'S FIELD EQUATIONS FOR STRESS-ENERGY TENSOR IN CLASSICAL TERMS

The compact form of Einstein's field equation [14] is defined as

$$G_{\mu\nu} = S^4 = \frac{2}{r^2} = \frac{8\pi G}{c^4} T_{\mu\nu} = \kappa T_{\mu\nu}, \qquad (1)$$

where $G_{\mu\nu}$ is the Einstein tensor, S^4 is the scalar curvature in four-dimensions, r is the radius of sphere, c is the speed of light in vacuum, κ is the so-called Einstein gravitational constant, with G being Newton's constant, $T_{\mu\nu}$ is the stress-energy tensor and indexes μ, ν run 1, 2, 3, 4.

According to this equation, scalar curvature for a spherical surface in four-dimensional spacetime directly depends of the stress-energy tensor, which is multiplied by the constant κ.

In order to describe stress-energy tensor in classical terms, we consider from the classical mechanics that work is a form of energy (as mechanical energy) [15], that is force times distance, equal to the line integral of the (mechanical) force **F** along a path C, given by

$$W = \int_C \mathbf{F} \mathrm{d}r, \tag{2}$$

where W is the work and r is the distance (from the center in circular motion).

According to the GR, energy is described as an equivalent to the mass of a given body by square of speed of light. On the other hand, according to the Newton's second law, square of velocity (speed of light, in this case) is equivalent to acceleration by distance, and force equivalent to mass by acceleration. Thus, writing those equivalences for "velocity" as speed of light, hence

$$E_W = mc^2 = ma_c r = Fr \therefore F = ma_c, \tag{3}$$

where E_W is the energy given by the work to move a body of mass m to the square of speed of light, F is the force and a_c is the acceleration (related with speed of light).

Furthermore, a tensor is generally defined as stress. A stress field is generally a force per unit area. Then, stress-energy tensor described in classical terms can be defined as force F per unit area A [16], where from expression (3) for a spherical surface is giving by

$$T_{\mu\nu}^G = \frac{E_W}{rA_{\mu\nu}} = \frac{F}{A_{\mu\nu}} = \frac{Ma_c}{4\pi R^2}, \tag{4}$$

where M is the mass of a massive body. Replacing expression (4) in the Einstein's field equation (1), yields

$$S_G = \frac{2}{r^2} = \frac{8\pi G}{c^4} T_{\mu\nu}^G = \frac{8\pi G}{c^4}\left(\frac{Ma_c}{4\pi R^2}\right). \tag{5}$$

This solution represents scalar curvature in classical terms for a spherical surface that is curved in the region of a uniform gravitational system given by a central force M (as that of the Sun) [17].

III. SOLUTION OF EINSTEIN'S FIELD EQUATIONS FOR MAGNETIC STRESS TENSOR IN CLASSICAL TERMS

On the other hand, "magnetic stress tensor" [18] is given from the magnetic force. In the same way, we can define magnetic stress tensor as the force (Lorentz force $\mathbf{F}_L$ for speed of light, in this case) per unit area, where for a spherical surface is giving by

$$T_{\mu\nu}^M = \frac{\mathbf{F}_L}{A_{\mu\nu}} = \frac{qc\times\mathbf{B}}{4\pi R^2} = \frac{q\mathbf{B}}{4\pi R^2 \mu_0 \varepsilon_0 c} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0 c}, \tag{6}$$

where μ_0 is the permeability of free space, ε_0 is the permittivity of free space, **E** is the electric field and the Poynting vector [19] is included, given by

$$S_e = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}. \tag{7}$$

If the field is only magnetic some terms are reduced, expression (6) becomes

$$T_{\mu\nu}^M = \frac{1}{\mu_0 c}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}\mathbf{B}_\mu \mathbf{B}_\nu = \frac{\mathbf{B}^2}{\mu_0}, \tag{8}$$

where $\mathbf{B}^2 = B_x^2 + B_y^2 + B_z^2$, which is a simplified equivalent expression of the magnetic term in the Maxwell stress tensor [7], defined as

$$T_{\mu\nu}^M = \frac{1}{\mu_0}\left(\mathbf{B}_\mu \mathbf{B}_\nu - \frac{\mathbf{B}^2}{2}\delta_{\mu\nu}\right), \tag{9}$$

where $\delta_{\mu\nu}$ is Kronecker's delta. It is proportional to the magnetic tension force [20], which is actually a pressure gradient and also a force density ($\mathrm{N/m^3}$) that acts parallel to the magnetic field.

Replacing expression (8) in (1), scalar curvature for solution of Einstein's field equations with a magnetic stress tensor in classical terms can be written as

$$S_M = \frac{2}{r^2} = \frac{8\pi G}{c^4} T_{\mu\nu}^M = \frac{8\pi G}{c^4}\left(\frac{\mathbf{B}^2}{\mu_0}\right). \tag{10}$$

An equivalent expression in terms of electric field can be derived by developing expression (10) in terms of permittivity of free space and considering equivalence between electric and magnetic fields given by $\mathbf{E} = c\mathbf{B}$, and reducing yields

$$S_E = \frac{2}{r^2} = \frac{8\pi G}{c^4} T_{\mu\nu}^E = \frac{8\pi G}{c^4}\left(\varepsilon_0 \mathbf{E}^2\right), \tag{11}$$

where $\mathbf{E}^2 = E_x^2 + E_y^2 + E_z^2$, being this tensor a simplified equivalent expression of the electric term in the Maxwell stress tensor.

IV. SOLUTION OF EINSTEIN'S FIELD EQUATIONS FOR AN ACCELERATED MAGNETIC SPHERICAL SURFACE

Considering a solution that combines both, stress-energy tensor given in classical terms and magnetic stress tensor given by the magnetic tension force [21], we can introduce magnetic field term in expression (5) by considering the equivalence with the inverse of spherical surface from the magnetic field equation with respect to a charge in motion, hence

$$\mathbf{B} = \frac{\mu_0 qc}{4\pi R^2} \therefore \frac{\mathbf{B}}{\mu_0 qc} = \frac{1}{4\pi R^2}, \qquad (12)$$

and then, replacing expression (12) in (4), yields

$$T_{\mu\nu}^{GM} = \frac{Ma_c}{4\pi R^2} = \frac{Ma_c \times \mathbf{B}}{\mu_0 qc}, \qquad (13)$$

which is a "magnetic stress-energy tensor" that combines both, stress-energy tensor and magnetic stress tensor, where cross product shows that direction of acceleration is perpendicular to the magnetic field. According to expression (8), it is when mechanical force equals Lorentz force ($\mathbf{F} = \mathbf{F}_L$), and also considering equivalences in expression (3), yields

$$Ma_c = \frac{Mc^2}{r} = qc \times \mathbf{B}, \qquad (14)$$

and then,

$$Mc^2 = (qc \times \mathbf{B})r = \mathbf{F}_L r = E_L, \qquad (15)$$

where E_L is the energy given by the magnetic field to move a charge to the square of speed of light. Simplifying speed of light term in both sides of expression (15), hence

$$Mc = (q\mathbf{B})r \therefore r = \frac{Mc}{q\mathbf{B}}, \qquad (16)$$

where r is an equivalent to the well-known gyro-radius expression (also known as radius of gyration, Larmor radius or cyclotron radius) [22] for a particle moving within a magnetic field.

Thus, replacing expression (13) in the expression (1) scalar curvature for Einstein's field equations with the magnetic stress-energy tensor is proposed as

$$S_{GM} = \frac{2}{r^2} = \frac{8\pi G}{c^4} T_{\mu\nu}^{GM} = \frac{8\pi G}{c^4}\left(\frac{Ma_c \times \mathbf{B}}{qc\mu_0}\right), \qquad (17)$$

which describe scalar curvature of a magnetic spherical surface in accelerated dilation that can be curved by a massive mass M.

In the same way, an equivalent expression in terms of electric field can be derived by developing expression (17) in terms of permittivity of free space and considering equivalence between electric and magnetic fields, and reducing yields

$$S_{GE} = \frac{2}{r^2} = \frac{8\pi G}{c^4} T_{\mu\nu}^{GE} = \frac{8\pi G}{c^4}\left(\frac{Ma_c \varepsilon_0 \mathbf{E}}{q}\right). \qquad (18)$$

It is noticed that expression (17) is analogous to the expression that results of consider a magnetic field **B** at parallel to a spherical surface in accelerated dilation. Development of this analogy is shown in Appendix A.

V. SOLUTION OF EINSTEIN'S FIELD EQUATIONS FOR AN ACCELERATED MAGNETIC WAVE

As known, wave equation can be written as a partial differential equation that describes the evolution of a wave over time in a medium where the wave propagates at the same speed independent of wavelength and independent of amplitude [23]. Thus, for the electric field **E** wave equation is defined as

$$\nabla^2 \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad (19)$$

and for the magnetic field **B**, wave equation in three dimensions is given by

$$\nabla^2 \mathbf{B} = \frac{\partial^2 \mathbf{B}}{\partial x^2} + \frac{\partial^2 \mathbf{B}}{\partial y^2} + \frac{\partial^2 \mathbf{B}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}. \qquad (20)$$

Considering the component in one-dimension of a magnetic wave in propagation along a single line toward the z-axis direction, expression (20) is reduced as

$$\frac{\partial^2 \mathbf{B}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}. \qquad (21)$$

This equation has a sinusoidal solution considering that magnetic field is in propagation as described by a wave function, which describes the displacement of particles in a wave as a function of time and their positions, hence

$$\mathbf{B} = y(z,t), \tag{22}$$

where y is displacement at z, t. An equivalent to the wave form is given by the proportionality between velocity of propagation and angular frequency, defined as

$$c^2 = \frac{\omega^2}{k^2}, \tag{23}$$

where ω is the angular frequency and k is the wave number. Thus, from expression (22) and (23), we can write expression (21) as

$$\frac{\partial^2 y(z,t)}{\partial z^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y(z,t)}{\partial t^2}. \tag{24}$$

Solving expression (24), as a function of their positions, yields

$$\frac{\partial^2 y(z,t)}{\partial z^2} = -k^2 A\cos(kz-\omega t) = -k^2 y(z,t), \tag{25}$$

where A is the wave amplitude; and solving as a function of time, hence

$$\begin{aligned}\frac{\partial^2 y(z,t)}{\partial t^2} &= -\omega^2 A\cos(kz-\omega t),\\ \frac{\partial^2 y(z,t)}{\partial t^2} &= -\omega^2 y(z,t) = a_y(z,t),\end{aligned} \tag{26}$$

where a_y is the acceleration in the y direction. Then, velocity is given by

$$\frac{\partial y(z,t)}{\partial t} = -\omega A sen(kz-\omega t) = v_y(z,t), \tag{27}$$

where v_y is the velocity in the y direction. Thus, solving expression (27) and according to expression (22), wave function in sinusoidal form for the magnetic field is given by

$$\mathbf{B} = y(z,t) = A\sin(kz-\omega t). \tag{28}$$

This expression shows the wave form propagation in sinusoidal form of the magnetic field as a harmonic sinusoidal wave. Simplifying for a simple harmonic motion as a function of time, hence

$$\mathbf{B} = y(t) = A\sin(\omega t) = A\sin\left(\frac{2\pi t}{T}\right), \tag{29}$$

where T is the period of oscillation.

When amplitude is related by the maximum amount of magnetic field, harmonic sinusoidal wave solution (29) can be written as $\mathbf{B} = B\sin(\omega t)$. Thus, replacing in expression (17) becomes

$$S_{GM} = \frac{8\pi G}{c^4}T^{GM}_{\mu\nu} = \frac{8\pi G}{c^4}\left(\frac{Ma_c B\sin(\omega t)}{qc\mu_0}\right), \tag{30}$$

which describe scalar curvature for an accelerated magnetic wave that can be curved by a massive mass M, with the magnetic stress-energy tensor expressed in sinusoidal form. Expression (30) in function of time can be written as

$$S_{GM} = \frac{8\pi G}{c^4}T^{GM}_{\mu\nu} = \frac{8\pi G}{c^4}\left(\frac{MB\sin(\omega t)}{q\mu_0 t}\right). \tag{31}$$

VI. CONCLUSIONS

This work aims to propose some solutions for the stress-energy-tensor and a magnetic stress tensor expressed terms of classical mechanics and electromagnetism, respectively. Then, those tensors are combined to attempt unify both; stress-energy tensor and magnetic stress-energy tensor, in the proposed magnetic stress-energy tensor. Then, considering the wave form of the magnetic field, it is proposed a solution of Einstein's Field Equations for an accelerated magnetic wave.

Regarding to the education, general theory of relativity is revisited describing the main concepts of this theory and some of the proposed exact solutions defined by the metrics of spacetime and the stress-energy tensor, where it is showed the possibility to apply some of the known equivalences to consider another possible properties from the classical theories.

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REFERENCES

[1] Einstein, A., *The Foundation of the General Theory of Relativity*, Annalen der Physik 49, 284–337 (1916).

[2] Stephani, H. et al, *Exact Solutions of Einstein's Field Equations*, 2nd ed. (Cambridge University Press, UK, 2003), pp. 422–483.

[3] Negi, P. S., *Exact Solutions of Einstein's Field Equations*, Int. J. Theor. Phys. **45**, 1684-1702 (2006). [arXiv:gr-qc/0401024].

[4] Guo, H., *On Principle of Inertia in Closed Universe*, Phys. Lett. **B653**, 88-94 (2007), p. 3 [arXiv:hep-th/0611341].

[5] Ryusuke, N., Chul-Moon, Y., Ken-ichi, N., *Evolution of density perturbations in large void universe*, Phys. Rev. D **85**, 103511 (2012) [arXiv:1202.1582].
[6] Lemoine, M., Lubo, M., Martin, J. and Uzan, J-P., *The stress-energy tensor for trans-Planckian cosmology*, Phys. Rev. D **65**, 023510 (2002). [arXiv:hep-th/0109128].
[7] Bimonte, G., Calloni, E. and Rosa, L., *The Gravity of Magnetic Stresses and Energy*, Phys. Rev. D **77**, 044026 (2008). [arXiv:0707.1077].
[8] Maxwell, J. C., *A Treatise on Electricity and Magnetism*, Vol. II, 3rd Ed. (Oxford University Press, UK, 1904), pp. 178-189.
[9] Faber , J. A. and Rasio, F. A., *Binary Neutron Star Mergers*, Living Rev. Relativity **15**, 57–60 (2012).
[10] Mashhoon, B., McClune, J. C. and Quevedo, H., *The gravitoelectromagnetic stress-energy tensor*, Class. Quant. Grav. **16**,1137-1148 (1999). [arXiv:gr-qc/9805093].
[11] Navarro, J. and Sancho, J. B., *A characterization of the electromagnetic stress-energy tensor*, Proceedings of the Spanish Relativity Meeting 2008. AIP Conference Proceedings **1122**, 360-363 (2009). [arXiv:1101.2505].
[12] Cornejo, A. G., *Maxwell-Faraday equation extension to four-dimensions*, Lat. Am. J. Phys. Educ. **5**, 717-720 (2011).
[13] Mo, T. C., *Electromagnetic Wave Propagation in a Uniformly Accelerated Simple Medium*, Radio Sci. **6**, 673–679 (1971).
[14] Einstein, A., *The Meaning of Relativity*, 5th ed. (Princeton University Press, Princeton, NJ, 1956), pp. 157-159.
[15] Resnick, R. and Halliday, D., *Physics*, Vol. I, (Wiley International Edition, 1966), pp. 124–165.
[16] Chen, W. and Han, D., *Plasticity for structural engineers*, (J. Ross Publishing, Florida, USA, 2007), pp. 46–71.
[17] Cornejo, A. G., *The effect of gravity hypothesis*, Lat. Am. J. Phys. Educ. **5**, 697-701 (2011).
[18] García-Duque, C. H. and García-Reyes, G., *General Magnetized Weyl Solutions: Disks and Motion of Charged Particles*, Gen. Relativ. Gravit. **43**, 3001 (2011). [arXiv:1009.1084].
[19] Richter, F., Florian, M. and Henneberger, K., *Poynting's theorem and energy conservation in the propagation of light in bounded media*, Europhys, Lett. **84**, 67005, 2-3 (2008). [arXiv:0710.0515].
[20] Symon, K. R., *Mechanics*, 3rd Ed. (Addison-Wesley Publishing Company, Boston, Massachusetts, USA, 1971).
[21] Chen, F. F., *Introduction to Plasma Physics and Controlled Fusion*, Plasma Physics, 2nd Ed. Vol. 1 (Plenum Press, New York, USA, 1984), p. 208.
[22] Tan, A., *Theory of Orbital Motion* (World Scientific Publishing Co. Pte. Ltd., Singapore, 2008), pp. 26-44.
[23] Purcell, E. M., *Electricity and Magnetism*, 2nd ed. (McGraw-Hill, New York, 1985), pp. 173-189.

APPENDIX A

From the magnetic field given in expression (12), it is found charge by velocity of light for a particle in motion, hence

$$\mathbf{B}=\frac{\mu_0 qc}{4\pi R^2}\therefore qc=4\pi R^2\frac{\mathbf{B}}{\mu_0}. \tag{A1}$$

Now considering that the spherical surface is in accelerated radial dilation, we can introduce acceleration of the spherical surface by multiplying square of acceleration in both terms of expression (A1); reordering and then considering classical equivalences for gravity with the accelerated circular motion [15], given by

$$v^2 r = ar^2 = GM \therefore a=\frac{GM}{r^2}, \tag{A2}$$

then, expression (A1) can be written as

$$\begin{aligned} qc &= 4\pi R^2\frac{a^2}{a^2}\frac{\mathbf{B}}{\mu_0}=4\pi R^2\frac{a^2 r^2}{c^4}\frac{\mathbf{B}}{\mu_0},\\ qc &= 4\pi R^2\frac{GMa}{c^4}\frac{\mathbf{B}}{\mu_0}, \end{aligned} \tag{A3}$$

and, multiplying expression (A3) by 2 and reordering, hence

$$\frac{2}{R^2}=\frac{8\pi G}{c^4}\left(\frac{Ma\mathbf{B}}{qc\mu_0}\right), \tag{A4}$$

which is an equivalent expression of (17), where as in the case, it is equivalent to the scalar curvature of a magnetic spherical surface in accelerated dilation that can be curved by a massive mass M.