Maximizing the Range of a Projectile from Takeoff Ramp

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Abstract

We discuss the motion of a projectile from takeoff ramp. Consider an object with a constant initial speed at the ground level moves up a takeoff ramp, with decreasing its speed due to gravity. It is launched from the top of the ramp, and after flying through the air, it lands on the ground. In this case, higher (lower) takeoff ramp leads to smaller (larger) launch speed, for given angle θ of incline of the takeoff ramp. Our problem is to find the optimal takeoff ramp which maximizes the range of the projectile for given θ. One finds that $\theta = 30°$ is critical; the takeoff ramp can enhance the range only for $\theta < 30°$. This problem is suitable for undergraduate students in calculus-based physics courses.

Keywords: Classical Mechanics, Projectile Motion, Optimization, Physics Education.

Resumen

Discutimos el movimiento de un proyectil desde la rampa de despegue. Considere que un objeto con una velocidad inicial constante a nivel del suelo se mueve hacia arriba por una rampa de despegue, y su velocidad disminuye debido a la gravedad. Se lanza desde lo alto de la rampa y, tras volar por los aires, aterriza en el suelo. En este caso, una rampa de despegue más alta (más baja) conduce a una velocidad de lanzamiento más pequeña (más grande), para un ángulo dado θ de inclinación de la rampa de despegue. Nuestro problema es encontrar la rampa de despegue óptima que maximice el alcance del proyectil para θ dado. Uno encuentra que θ = 30° es crítico; la rampa de despegue puede mejorar el alcance solo para θ <30°. Este problema es adecuado para estudiantes de pregrado en cursos de física basados en cálculo.

Palabras clave: Mecánica clásica, Movimiento de proyectiles, Optimización, Educación física.

I. INTRODUCTION

Discussing the motion of a projectile is one of the most standard topics in introductory physics courses. In this problem, when the projectile is launched and lands at the same height, one finds that the horizontal range of the projectile is a maximum when the launching angle above the horizontal is 45°. It is discussed that if the launching point is located at some height above the horizontal, the optimal angle is no longer 45° [1]. Variations of this problem, such as the effects of air resistance [2], a projectile shot along a slope [3], solving method without calculus [4], discussions by using time of flight [5], the case of ski jump with a linear landing hill [6], have been studied by many authors. In Ref. [7], the minimum launch speed to hit a target above the horizontal is discussed.

In this paper, we discuss the motion of a projectile launched from takeoff ramp. Let us consider that an object has an initial speed at the ground level and it moves up a takeoff ramp with decreasing its speed due to gravity, and finally it is launched from the takeoff ramp. The projectile flies through the air, and it lands at a point on the ground. In this motion, if the initial speed at the ground level is constant, the launch speed from the takeoff ramp depends on the height of the ramp. This means that higher (lower) takeoff ramp leads to smaller (larger) launch speed, for given angle of the takeoff ramp. Our problem is to find the optimal takeoff ramp to maximize the range of the projectile, and to find the condition that takeoff ramp enhances the range of the projectile relative to that of ordinary projectile motion. This optimization problem is suitable for undergraduate students in calculus-based physics courses.

II. PROJECTILE FROM TAKEOFF RAMP

Figure 1 shows a schematic diagram of a projectile from takeoff ramp. Let us consider that a point-like object has the initial speed v_0 at time $t = 0$ at the origin. It moves along a frictionless incline of a takeoff ramp, which makes an angle of θ above the horizontal and a length of ℓ. It is launched from point P with a speed v_P at time t_P, and after flying through the air due to gravity, it lands at point Q at time t_Q. The position of point P and Q is $(x_P, y_P) = (\ell \cos\theta, \ell \sin\theta)$ and $(x_Q, 0)$, respectively, and we define the flying distance L as $L = x_Q - x_P$.

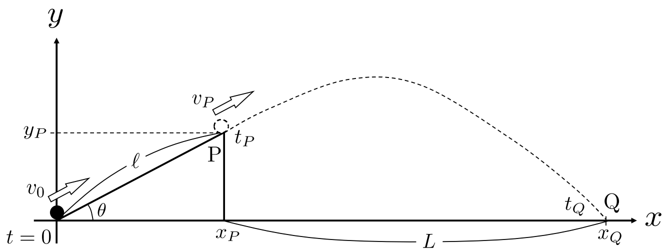

FIGURE 1. Schematic diagram of a projectile from takeoff ramp.

At point P, v_P and t_P is given by

$$v_P = \sqrt{v_0^2 - 2g\ell \sin\theta}, \tag{1}$$
$$t_P = \frac{v_0 - v_P}{g \sin\theta}, \tag{2}$$

where g is the gravitational acceleration. Hereafter we suppose $\ell \le v_0^2/(2g\sin\theta)$, the object always flies out of the ramp.

After flying through the air, the projectile lands at point Q, with

$$t_Q = t_P + \frac{1}{g}\left[v_P \sin\theta + \sqrt{(v_P \sin\theta)^2 + 2g\ell \sin\theta}\right], \tag{3}$$

$$x_Q = x_P + v_P \cos\theta \left(t_Q - t_P\right). \tag{4}$$

As a consistency check, when $\ell \to 0$, one can verify that Eq.(4) leads to the well-known results

$$x_Q(\ell = 0) = L_0 = \frac{v_0^2}{g}\sin 2\theta. \tag{5}$$

Now let us consider the optimization condition of the flying distance $L = x_Q - x_P$,

$$L = \frac{\cos\theta}{g} v_P [v_P \sin\theta + A], \tag{6}$$

where

$$A = \sqrt{(v_P \sin\theta)^2 + 2g\ell \sin\theta}. \tag{7}$$

For given θ, the extremum condition for L is calculated from its derivative in terms of ℓ. Thus

$$\begin{aligned}\frac{dL}{d\ell} &= \frac{dv_P}{d\ell}\cdot\frac{dL}{dv_P} \\ &= \frac{dv_P}{d\ell}\cdot\frac{\cos\theta}{gA}(2v_P A\sin\theta + v_0^2 - 2v_P^2\cos^2\theta).\end{aligned} \tag{8}$$

From the extremum condition for L, $dL/d\ell = 0$, the parenthesis in Eq. (8) must vanish. By squaring the $(9)cos$,

$$(2v_P A \sin\theta)^2 = (-v_0^2 + 2v_P^2\cos^2\theta)^2, \tag{9}$$

gives

$$v_P = \frac{v_0}{\cos\theta}\sqrt{\frac{1+\sin\theta}{2}}, \tag{10}$$

where $v_P \ge 0$. The other solution of Eq. (9), $v_P = v_0/\cos\theta\sqrt{(1-\sin\theta)/2}$, is not appropriate because it does not satisfy the original equation $dL/d\ell = 0$. This inappropriate solution arises from squaring the extremum condition, that is, Eq. (9).

Substituting Eq. (10) into Eq. (1), we obtain the optimal length of the takeoff ramp

$$\ell_{max} = \frac{v_0^2}{g}\frac{\sin\theta}{(\sin 2\theta)^2}(\cos 2\theta - \sin\theta). \tag{11}$$

When the takeoff ramp has length of ℓ_{max}, by substituting Eqs. (10) and (11) into Eq. (6), we obtain the maximal value of the flying distance,

$$L_{max} = \frac{v_0^2}{g}\frac{1+\sin\theta}{2\cos\theta}. \tag{12}$$

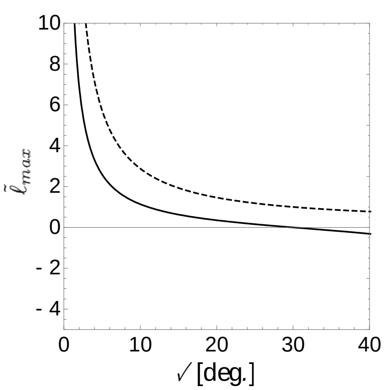

FIGURE 2. Relation of $\tilde{\ell}_{max}$,defined in the main text, and θ from Eq. (11) (solid), and its upper bound $1/(2\sin\theta)$ (dashed).

Figure 2 shows relation of a dimensionless parameter $\tilde{\ell}_{max} = \ell_{max}/(v_0^2/g)$ and θ expressed in Eq. (11) (solid curve), and upper bound of $\tilde{\ell} = \ell/(v_0^2/g)$, $1/(2\sin\theta)$ as discussed below Eq.(2) (dashed curve). As seen from the figure, larger $\tilde{\ell}_{max}$ is monotonically decreasing function of θ. It vanishes at $\theta = 30°$, which corresponds to a solution of

$$\cos 2\theta - \sin\theta = 0, \tag{13}$$

derived from Eq. (11). The fact that $\tilde{\ell}_{max}$ becomes negative for $\theta > 30°$ tells us that takeoff ramp can enhance the flying distance only for $\theta < 30°$.

To see this, we plot the flying distance as a function of θ. Figure 3. shows $\tilde{L}_{max} = L_{max}/(v_0^2/g)$(solid curve) and $\tilde{L}_0 = L_0/(v_0^2/g)$(dashed curve). As seen from the figure, $\tilde{L}_{max} > \tilde{L}_0$ is satisfied for $\theta < 30°$, and $\tilde{L}_{max} = \tilde{L}_0$ at $\theta =$

$30°$. Since the takeoff ramp disappears for $\theta > 30°$, the curves in this region has no physical meanings (shaded in the figure). We define the enhancement of the flying distance, $\Delta\tilde{L}$, from that of ordinary projectile motion as

$$\Delta\tilde{L} = \tilde{L}_{max} - \tilde{L}_0 = \frac{2(1+\sin\theta)}{\cos\theta}\left(\sin\theta - \frac{1}{2}\right)^2. \quad (14)$$

Red dotted curve in Figure 3 represents $\Delta\tilde{L}$, and $\Delta\tilde{L} \geq 0$ for $\theta < 30°$, and it has its minimal value $\Delta\tilde{L} = 0$ when $\theta = 30°$, as expected. An advantage of takeoff ramp is that the object has its initial height $\ell \sin\theta$ at point P, although its launching speed v_P becomes smaller than v_0. Launching from the takeoff ramp is more effective to enhance the flying distance for smaller θ, and such a benefit of the takeoff ramp is dissipated at $\theta = 30°$.

As an alternative way to find $\theta = 30°$, one can derive the condition to enhance the flying distance from Eq. (6) without differentiation. Define $\Delta L' = L - L_0$. From the condition for $\Delta L' > 0$ and after some computation, one finds

$$\ell < \frac{v_0^2}{g} \cdot \frac{1 - 4\sin^2\theta}{2\sin\theta}. \quad (15)$$

Since $\ell > 0$ by definition, $\theta < 30°$ must be satisfied. And when Eq. (15) is satisfied, $\Delta L' > 0$, that is, the flying distance is enhanced by takeoff ramp relative to that of the case without takeoff ramp.

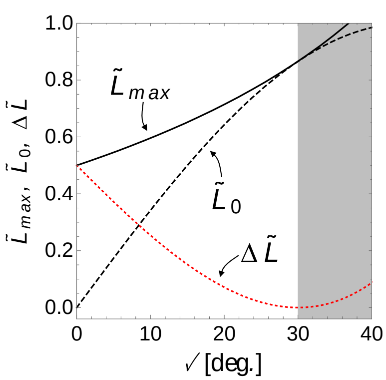

FIGURE 3. Parameters $\tilde{L}_{max}$, $\tilde{L}_0$, and $\Delta\tilde{L}$, defined in the main text, as a function of θ. The shaded region $\theta > 30°$ is excluded.

III. CONCLUSIONS

We have discussed the motion of a projectile from a takeoff ramp. Since higher (lower) takeoff ramp leads to smaller (larger) launch speed, for given angle of the takeoff ramp, the range of the projectile nontrivially depends on the size of the takeoff ramp. By solving the extremum condition for the range of the projectile, we found that $30°$ of the launching angle θ is critical; for $\theta < 30°$, one can find the optimal size of the takeoff ramp to maximize the range of the projectile. In this case, the range is larger than that of the case without takeoff ramp. On the other hand, for $\theta > 30°$, takeoff ramp never enhances its range. This optimization problem is a good exercise for undergraduate students in calculus-based physics courses.

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