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The Motion of a Hoop with an Attached Mass Causes to Jump

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Abstract

This study investigates the phenomenon of a hoop with an attached mass exhibiting a jumping motion when rolling with sufficient angular momentum. It is explained by the interplay of angular momentum, gravitational force, and the rotational dynamics of the system. Key parameters influencing this behavior include the mass of the hoop, the attached mass, and the initial angular velocity. The study explores whether similar behavior can be reproduced with alternative unbalanced mass distributions on a wheel and discusses the implications of varying ground profiles on the jumping motion. Results show the specific dynamics depend on the distribution and magnitude of the mass. Changes in ground profile are found to alter the trajectory and characteristics of the jumping motion and highlighting the importance of environmental factors in such dynamic systems. In our study, we explored the impact of three distinct hoop materials - metal, plastic, and wood - on the dynamics of a jumping motion. Alongside material variation, we investigated the influence of hoop size and the mass attached to the hoop. Through experimental analysis, this article elucidates how these parameters shape the jumping phenomenon. By examining the experimental results, the analysis of the results presented in this study utilized COMSOL for simulation purposes, with MATLAB employed for generating phase diagrams. Additionally, TRACKER software facilitated the numerical analysis, particularly in determining the jump angle. By leveraging these tools, we were able to comprehensively examine the experimental data and draw the conclusions regarding the influence of hoop material, size variations, and attached mass on the dynamics of the jumping motion.

Keywords: Hoop, Jumping Motion, Rolling.

Resumen

Este estudio investiga el fenómeno de un aro con una masa adosada que realiza un salto al rodar con suficiente momento angular. Este fenómeno se explica por la interacción entre el momento angular, la fuerza gravitacional y la dinámica rotacional del sistema. Los parámetros clave que influyen en este comportamiento incluyen la masa del aro, la masa adosada y la velocidad angular inicial. El estudio explora si se puede reproducir un comportamiento similar con distribuciones de masa desequilibradas alternativas sobre una rueda y analiza las implicaciones de las variaciones en el perfil del terreno sobre el salto. Los resultados muestran que la dinámica específica depende de la distribución y la magnitud de la masa. Se observa que los cambios en el perfil del terreno alteran la trayectoria y las características del salto, lo que subraya la importancia de los factores ambientales en este tipo de sistemas dinámicos. En nuestro estudio, exploramos el impacto de tres materiales distintos para el aro (metal, plástico y madera) en la dinámica del salto. Además de la variación del material, investigamos la influencia del tamaño del aro y la masa adosada. Mediante análisis experimental, este artículo explica cómo estos parámetros dan forma al fenómeno del salto. Para el análisis de los resultados experimentales presentados en este estudio, se utilizó COMSOL para la simulación y MATLAB para la generación de diagramas de fase. Además, el software TRACKER facilitó el análisis numérico, especialmente para determinar el ángulo de salto. Gracias a estas herramientas, pudimos examinar exhaustivamente los datos experimentales y extraer conclusiones sobre la influencia del material del aro, las variaciones de tamaño y la masa añadida en la dinámica del salto.

Palabras clave: Aro, Salto, Rodadura.

I. INTRODUCTION

For a while now, reports and observations of hopping hoops have been made. The phenomena that happens when a hoop jumps while rolling in a vertical plane by carrying a heavy object fixed to the rim is explained for the first time in this work (Fig. 1).

FIGURE 1. Schematic observation of jumping motion.

Originally described in "A Mathematician's Miscellany" by Littlewood [1] in 1953, the hypothetical problem of a rigid, massless hoop filled with a particle that rolls without slipping can be used as an engaging issue in the classroom for planar motion of rigid bodies.

Tokieda [2] revived the problem by basing his analysis on the geometric features of the motion.

Littlewood and Tokieda both come to the same conclusion—that once the particle is at the highest point along its cycloidal journey, the hoop will hop after rolling through 90°.

Since Tokieda's work, variations of this problem have drawn a lot of interest. Butler [3] and Theron [4] made an effort to disprove the possibility of this hop. However, as Theron and du Plessis argue in [5], the proof in both situations was predicated on false assumptions.

The genuinely singular challenge of a massless, rigid hoop loaded with a particle, which we have not been able to definitively demonstrate, does not hop when the normal reaction becomes zero.

Nevertheless, in [5], we incorporate practical friction coefficient values into the model and discover that a skimming motion, in which the massless hoop stays in contact with the surface despite the absence of any response force between the surface and the hoop, can occur after the rolling phase.

The motion of a hoop rolling in a vertical plane while carrying a heavy particle fastened to the rim is covered in Theron's study [6]. The hoop is not rigid, as was previously determined; instead, the elasticity of the system yields findings consistent with earlier reports of hula-hoop hopping.

The primary outcome of this investigation is the determination of the prerequisites needed for hopping to happen following a rotation of less than 90 degrees, beginning with the particle at its highest position.

This work is an effort to use ring elasticity to explain this observed occurrence. We analyze a very basic elastic model in which, among other things, we assume that the moment of inertia is invariant to ring deformation. Even with its simplicity, this methodology produces useful outcomes.

Also, after examining the theory of this phenomenon and performing calculations to find the maximum height, we will examine the results of the tests performed in different conditions.

II. THEORETICAL FRAMEWORK

The first theory that we will investigate here is the T.F. Tokieda's theory [2] for a weightless hoop with a point mass applied to it, which it is, said that it should be placed near the position of unstable equilibrium (Fig.2).

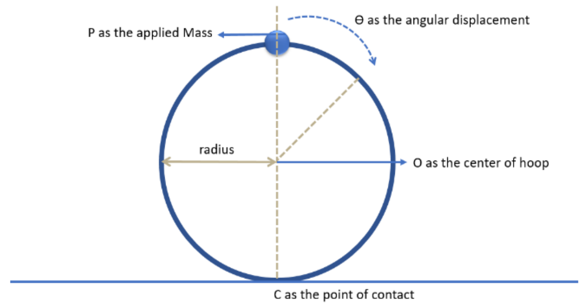


FIGURE 2. Schematic view of the hoop.

It can be said that the best way to prove the hoop indeed hops is to calculate the force that the hoop exerts against the floor at the point of contact, and to check that it changes to negative after the hoop has rolled π/2. But unfortunately it hardly explains "why the hoop should even hop?" although by the following description it is possible to see the different forces applied to the system (Fig. 3).

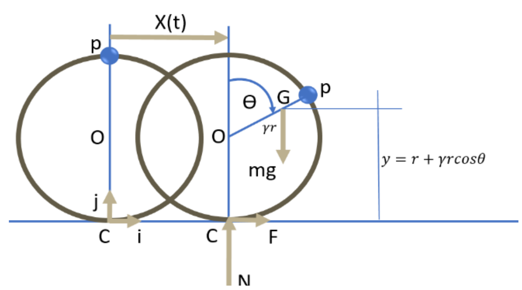


FIGURE 3. Forces applied to the system.

The following theory is based on the fact that the weight of the hoop is equal to zero in comparison with the point mass added to the hoop (it is not actually a weightless hoop but in comparison with the point mass it is negligible).

As the hoop will rotate down if we follow a line from the point mass on the position of unstable equilibrium it becomes a line coming down and it will stop on the ground and it will go up again (Fig. 4).

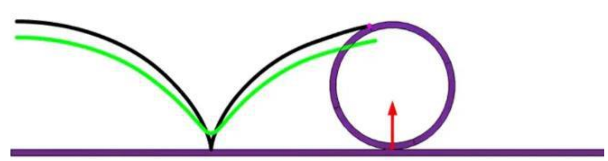


FIGURE 4. COMSOL simulation of the jumping hoop.

By this description we can say that there are to conflicted shapes, for sure there is a cycloid which form by following the point mass on the hoop, but also we are neglecting mass of the hoop so technically because the hoop is massless and a single point mass will form a parabola while hitting the ground, this means both shapes should be investigated to see in which parts of both graphs they will be aligned.

We plotted the both shapes of a cycloid and parabola that forms when the hoop rolls on the ground while there is a point mass attached to it using COMSOL and GEOGEBRA.

We can observe both cycloid and parabola that forms in both simulations which represents the displacement of the hoop, so the p stands for the point mass on the hoop and it is possible to see in the schematic view that when the point mass reaches to the π/2 of the hoop we can observe that the green graph which is the parabola will be outside the cycloid (black graph) (Fig. 5).

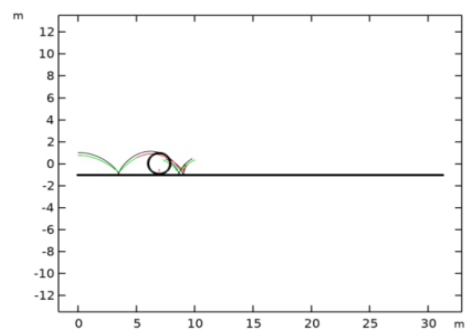


FIGURE 5. Displacement (mbd).

The moment that parabola is outside the cycloid is when the rolling motion will turn into a hopping motion (Fig. 6).

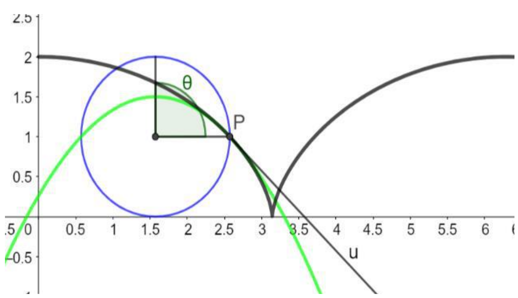


FIGURE 6. Jumping occurs while parabola is outside of the cycloid.

Now based on the T.F Tokieda Theory that represents:

$$\frac{m}{2}(\dot{x}^2+\dot{y}^2)+mgy=\frac{m}{2}v_0^2+\mathrm{mg}2R$$

Along the cycloid:

$$x(t)=R\theta(t)+Rsin\theta(t)$$

$$\mathrm{y}(t)=R+Rcos\theta(t)$$

1) M (point mass) presses the hoop down as long as the imagined parabola at p departs below the cycloid
2) M pulls the hoop up, and the hoop hops, as soon as the imagined parabola at p departs above the cycloid.

By construction the parabola and the cycloid have the same zeroth, therefore departure below or above will be decided by an inequality between their second derivatives. If we shove the point mass off the point of unstable equilibrium with initial velocity being equal to zero, with g being the gravitational acceleration and R the radius of the hoop as it is shown at (1), the conservation of theory dictates.

$$\frac{m}{2}\left[\left(R\dot{\theta}+Rcos\theta.\dot{\theta}\right)^2+\left(-Rsin\theta.\dot{\theta}\right)^2\right]+$$
$$mg(R+Rcos\theta) \qquad =\frac{m}{2}v_0^2+mg2R$$

$$\dot{\theta}^2=\frac{4gRsin^2\left(\frac{\theta}{2}\right)+v_0^2}{4R^2cos^2(\frac{\theta}{2})},\ \dot{y}=-Rsin\theta.\dot{\theta}$$
$$=-\sin(\frac{\theta}{2})\sqrt{4gRSin^2\left(\frac{\theta}{2}\right)+v_0^2}$$

Therefore, m pulls the hoop up, thereby making it hop, as soon as the second derivative of the parabola exceeds that of the cycloid; the hop occurs at minimal θ with

$$\ddot{y}=-2gsin^2\left(\frac{\theta}{2}\right)-\frac{v_0^2}{4R},$$
$$1)\ -g\geq\ \ddot{y}\big(\theta(t)\big)$$
$$2)\ \sin\left(\frac{\theta}{2}\right)\geq 1/\sqrt{2}(1-\frac{v_0^2}{4gR})^{1/2}$$

Especially for initial velocity being equal to zero the hoop hops at θ being equal to π/2.

By the information given in the theory it is possible to obtain the following diagrams from the simulation which is simulated by this theory (Figs. 7-10).

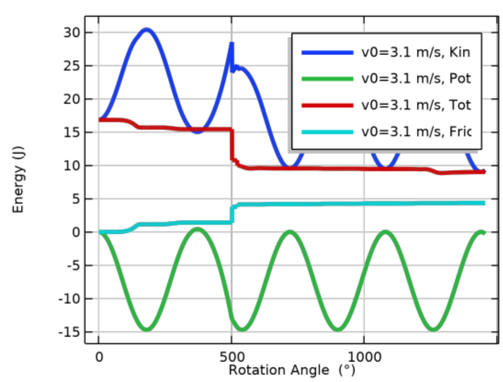


FIGURE 7. Energy vs Rotation angle diagram.

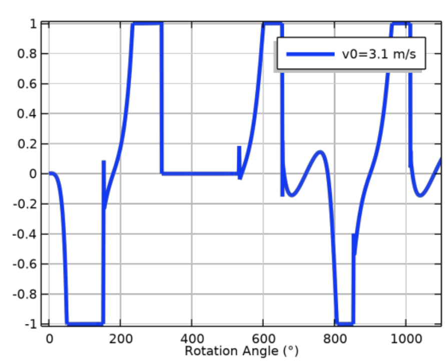


FIGURE 8. Velocity vs Rotation angle diagram.

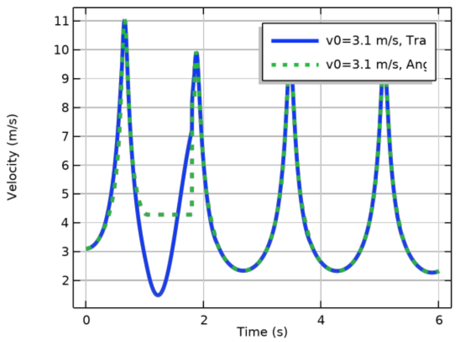


FIGURE9. Velocity vs Time diagram.

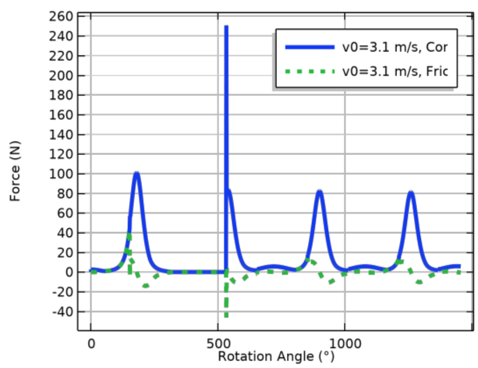


FIGURE10. Force vs Rotation angle diagram.

These graphs for force dependent and velocity dependent diagrams were investigated using simulation of a hopping hoop by COMSOL.

Also, we can express the potential energy that the potential energy of the system is only affected by the vertical position of the point mass, and can be written as:

$$W_P = \gamma m g R(\cos(\theta) - 1)$$

that when the point mass reaches its upper position, the potential energy becomes zero. Kinetic energy can be written as follows after some algebra which includes the location and speed of the center of gravity:

$$W_k = mR^2(1 + \gamma\cos(\theta))\dot{\theta}^2$$

Given the expressions for both types of energy, the principle of conservation of energy can be used to arrive at a closed expression for the angular velocity as a function of the angle of rotation:

$$W_K + W_P = const = W_K|_{\theta=0}$$

Inserting the kinetic and potential energy expressions, we get:

$$mR^2(1 + \gamma\cos(\theta))\dot{\theta}^2 =$$

$$-\gamma m g R(\cos(\theta) - 1) + m v_0^2(1 + \gamma)$$

Thus

$$\dot{\theta} = \frac{1}{R}\sqrt{\frac{\gamma g R(1 - \cos(\theta)) + v_0^2(1 + \gamma)}{1 + \gamma\cos(\theta)}}$$

When

$$\theta = \pi$$

then:

$$\dot{\theta}_{max} = \frac{1}{R}\sqrt{\frac{2\gamma g R + v_0^2(1 + \gamma)}{1 - \gamma}}$$

What happens as γ approaches 1 (the main problem with just a point mass). The maximum angular velocity approaches infinity! It looks non-physical [11].

III. MATERIALS AND METHODS

By using various masses we can observe the effect that different masses will have on the hop, it is said in the question that by changing the profile of the ground we might be able to see its effect on the way that the hoop will slide on the ground which will affect the hopping motion, in some investigations we can say that the point mass will lift the hoop and we will see the jumping motion also in some of the experiments we can see that the hoop will accelerate down without any jumping motion.

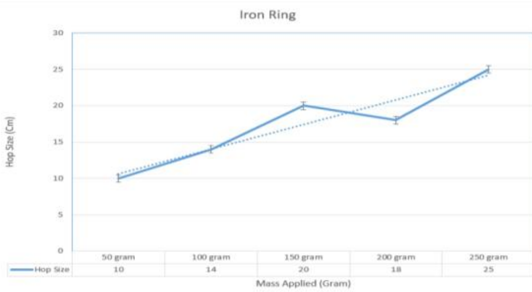
The experiments include the main factors to investigate how they will affect the phenomenon and what changes they cause in the size of the jumping motion.

The experimental setup includes hoops with different internal and external diameter to investigate how the diameter of the hoop will affect the phenomenon, 3 types of material for the hoop which covers a wooden, iron and plastic material for the hoop, with different point masses applied to the system with 50,100,150,200 and 250 gram mass (Fig. 11).

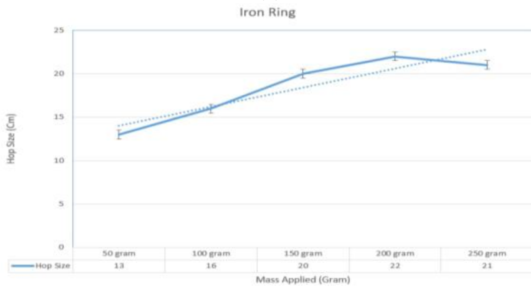
FIGURE 11. Different Point mass applied to the system.

IV. RESULTS AND DISCUSSION

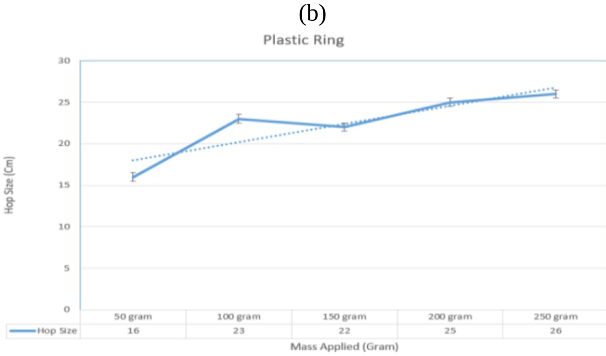
Sliding the hoop with zero initial velocity so we can have a comparison between the theoretical result and experimental result, we can obtain the main graph which is the size of the hop while attaching different point mass to the hoop, this experiment is investigated for 3 materials of the hoop and hoops with external diameter being equal to 30 cm, 45 and 55 cm (Figs.12a,b , c and d).



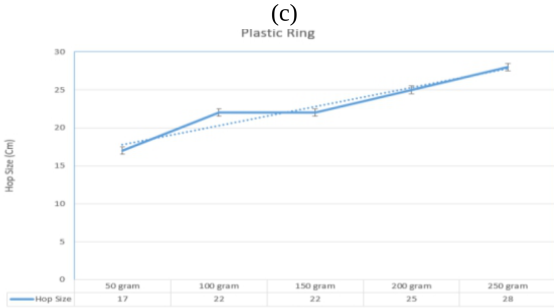
(a)



(b)



(c)



(d)

FIGURE 12. Size of the hop vs Point mass in different rings, (a) and (b) in Iron ring, (c) and (d) in Plastic ring.

The graph represents the change in size of the hop by adding different point mass to the metal hoop which we can observe there is an almost ascending line that proves the metal hoop will hop more if we add a greater mass (250 gram). The hoop itself weighs 56 grams using the hoop with 0 cm external diameter (Fig.12a) and 77 grams using the hoop with 45 cm external diameter (Fig.12b) which both are clearly not weightless but low weighted in comparison with the point mass.

The same experiment is investigated for plastic hoops with the same external diameters being equal to 30,45 (Fig.12c and d) and 55 cm, which the same ascending trend can be seen by applying a range of point mass from 50 to 250 grams to the plastic hoop and the size of the hop will increase by increasing the mass.

Therefore, by a comparison between theory and experiment it is possible to say that the experiment confirms the theory which represents that the jumping motion occurs in $\pi/2$ and by the experiment we were able to estimate the size of the hop that happens in $\pi/2$ using tracker by tracking the trace of point mass and measure the distance between the ground and the hoop while it hops (Fig.13).

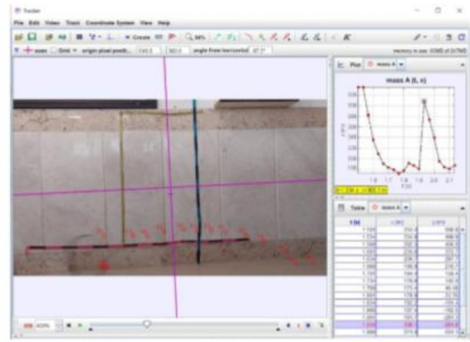


FIGURE 13. To estimate the size of the hopping by Tracker.

V. CONCLUSIONS

Theory and experiment in comparison proves that there is a direct relation between the point mass and the size of the hop, although comparing wooden, iron and plastic hoop we can observe that the data proves that the size of the jump in plastic hoop is more than both iron and wooden hoop which it is possible to confirm it by the main theory which is the hoop being weightless in comparison with the point mass and knowing that the plastic hoop with the same external diameter and width, weights lower than wooden and iron hoop means the theory can be more accurate for a hoop with a lower weight. (although the hopping happens in all three different materials we used in the experiment).

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