Does Fourier's Law of Heat Conduction Contradict the Theory of Relativity?

E. Marín
Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada, Instituto Politécnico Nacional. Legaria 694, Col. Irrigación, C.P. 11500, México D.F., México.

E-mail: emarin63@yahoo.es, emarinm@ipn.mx

(Received 7 April 2011; accepted 1 June 2011)

Abstract

This paper is about the reasons of why the well known Fourier´s Law of heat conduction is inconsistent with one of the main results of Einstein´s theory of relativity, namely that the greatest known speed is that of the electromagnetic waves propagation in vacuum. Simple (but on solid physical arguments constructed) modifications to the laws of heat conduction will be presented that help to overcome this apparent paradox. Some of their fields of applications will be also presented with the aim to introduce teachers and students dealing with heat transfer problems to these questions that are not often discussed in standard text books and courses.

Keywords: Heat transfer, Fourier's Law, relaxation time.

Resumen

Este artículo trata sobre las razones de por qué la ley de Fourier para la conducción del calor es inconsistente con uno de los principales resultados de la teoría de la relatividad de Einstein, aquel que plantea que la mayor velocidad conocida es la de la propagación de las ondas electromagnéticas en el vacío. Se presentarán modificaciones sencillas (pero construidas sobre argumentos físicos sólidos) a las leyes de la conducción del calor que ayudan a resolver esa aparente paradoja. Se mostrarán también algunos campos de aplicación con el objetivo de introducir a estudiantes y maestros interesados en fenómenos de transferencia de calor en esas cuestiones que no se discuten normalmente en los cursos y libros de texto estándar.

Palabras clave: Transferencia de calor, Ley de Fourier, tiempo de relajación.

PACS: 44.10.+I; 66.30.Xj; 66.70.+f **ISSN 1870-9095**

I. INTRODUCTION

The answer to the question giving title to this article is **yes**. Fourier's law of heat conduction predicts an infinite speed of propagation for thermal signals, i.e. a behavior that contradicts Einstein´s relativity theory. This theme is often overlooked in heat transfer text books and in engineering and physics courses. Therefore it is the aim of this paper to call the attention of teachers and students to it, explaining how the above mentioned paradoxical situation can be overcome using a simple model. Some fields of research will be highlighted where this model will be useful.

II. FOURIER´S LAWS

In 1822, Joseph Baptiste Fourier, a French scientist, pointed out his most famous work named "Analytical Theory of Heat", and proposed the famous law of heat conduction having his name [1]. The **(first) Fourier's Law** states that the time rate of heat transfer through a material is proportional to the negative gradient of temperature and to the area at right angles through which the heat flows. In differential form it lauds.

$$\vec{q} = -k\vec{\nabla}\mathrm{T}, \tag{1}$$

where q is the heat flux (W/m^2), k is the thermal conductivity and T is the temperature.

This is a very simple empirical law that has been widely used to explain heat transport phenomena appearing often in daily life, engineering applications and scientific research.

When combined with the law of energy conservation for the heat flux.

$$\frac{\partial E}{\partial t} = -\mathrm{div}(\vec{\mathrm{q}}) + \mathrm{Q}, \tag{2}$$

where Q represents the internal source of heat and $\partial E/\partial t = \rho c \partial T/\partial t$ is the temporal change in internal energy, E, for a material with density ρ and specific heat c, and assuming constant thermal conductivity, Fourier's law leads to another important relationship, namely the non-stationary heat diffusion equation also called **second Fourier's law** of conduction. It can be written as:

$$\nabla^2 T - \frac{1}{\alpha}\frac{\partial T}{\partial t} = -\frac{Q}{k}, \quad (3)$$

with $\alpha = k/\rho c$ as the thermal diffusivity.

III. FOURIER´S LAWS INCONSISTENCE

Consider for example a flat slab and apply at a given instant a supply of heat to one of its faces. Then according to Eq. (1) there is an instantaneous effect at the rear face. Loosely speaking, according to Eq. (1), and also due to the intrinsic parabolic nature of the partial differential Eq. (3), the diffusion of heat gives rise to infinite speeds of heat propagation. This conclusion, named by some authors the paradox of instantaneous heat propagation, is not physically reasonable because it clearly violates one important principle of the Einstein's special theory of relativity: the velocity of light in vacuum is the greatest known speed and has a finite value of 2.998×10^8m/s.

The above mentioned contradictions between the Fourier´s heat conduction laws and the theory of relativity can be overcome using several models. Because the most of them have been inspired in the so-called CV model, the attention of this paper will be focused on it.

IV. THE CATTANEO-VERNOTTE MODEL

This model takes its name from the authors of the two pioneering works on this subject, namely that due to Cattaneo [2] and that developed later and (apparently) independently by Vernotte [3]. The CV model introduces the concept of the relaxation time, τ, as the build-up time for the onset of the thermal flux after a temperature gradient is suddenly imposed on the sample.

Suppose that as a consequence of the temperature existing at each time instant, t, the heat flux appears only in a posterior instant, $t + \tau$. Under these conditions Fourier's Law adopts the form:

$$q(x, t+\tau) = -k\,\frac{\partial T(x,t)}{\partial x}. \quad (4)$$

If τ is small (as it should be, because otherwise the first Fourier´s law would fail when explaining every day phenomena), then we can expand the heat flux in a Taylor Serie around $\tau = 0$ obtaining:

$$q(x, t+\tau) = q(x,t) + \tau\,\frac{\partial q(x,t)}{\partial t}, \quad (5)$$

where we neglected higher order terms.

Substituting Eq. (5) in Eq. (4) leads to the modified Fourier´s law of heat conduction or **CV equation** that states:

$$\tau\frac{\partial \vec{q}}{\partial t} + \vec{q} = -k\vec{\nabla}\mathrm{T}. \quad (6)$$

Here the time derivative term makes the heat propagation speed finite. Eq. (6) tells us that the heat flux does not appear instantaneously but it grows gradually with a build-up time, τ. For macroscopic solids at ambient temperature this time is of the order of 10^{-11}s so that for practical purposes the use of Eq. (1) is adequate, as daily experience shows.

Substituting Eq. (6) into the energy conservation law (Eq. (2)) one obtains [4]:

$$\nabla^2 T - \frac{1}{\alpha}\frac{\partial T}{\partial t} - \frac{1}{u^2}\frac{\partial^2 T}{\partial t^2} = -\frac{1}{k}\left(Q + \tau\frac{\partial Q}{\partial t}\right). \quad (7)$$

Here $u = (\alpha/\tau)^{1/2}$ represents a (finite) speed of propagation of the thermal signal, which diverges only for the unphysical assumption of $\tau = 0$.

Eq. (7), that we will call the **second CV law**, is a hyperbolic, telegraph like equation, instead of a parabolic Eq. (3). Hence a wave nature of heat propagation is implied and new phenomena can be advised.

V. FIELDS OF APPLICATION

The CV model, although necessary, has some disadvantages, among them: i). The hyperbolic differential Eq. (7) is complicate from the mathematical point of view and in the majority of the physical situations has non analytical solutions. ii) The relaxation time of a given system is in general an unknown variable. Therefore care must be taken in the interpretation of its results. Nevertheless, several examples can be found in the literature.

As described with more detail by Joseph and Preziosi [5], Band and Meyer [6], in the same year in which Cattaneo's work appeared, and Osborne [7], two years later, proposed exactly the same Eq. (7) to account for dissipative effects in liquid He II, where temperature waves propagating at velocity u were predicted [8, 9, 10] and verified. Due to these early works the speed u is often called the second sound velocity. More recently Tzou reported on phenomena such as thermal wave resonance [11] and thermal shock waves generated by a moving heat source [12]. Very rapid heating processes must be explained using the CV model too, such as those taking place during the absorption of energy coming from ultra short laser pulses [13] and during the gravitational collapse of some stars [14], as many numerical and analytical calculations indicate.

On the other hand, one actual field of rapid development is nanoscience and nanotechnology, in which several studies have reported nanoscale heat transfer behavior deviating significantly from that at the normal scale [15]. It is well known that thermal time constants, τ_c, characterizing heat transfer rates depend strongly on particle size and on its thermal diffusivity, α. One can assume that for spherical particles of radius L, these times scale proportional to L^2/α [16]. As for condensed matter the order of magnitude of α is $10^{-6}m^2/s$, for spherical particles having nanometric diameters, says for example between 100 and 1 nm, we obtain for these times values ranging from about 10 ns to 1ps, which are very close to the above mentioned relaxation times. At these short time scales Fourier's laws do not work in their initial forms. In this field and for continuous transient heating the work of Vadasz *et al.* [17] is illustrative. On the basis of theoretical calculations and experimental data these authors demonstrated that the hyperbolic heat transfer could have been the cause of the extraordinary heat transfer enhancement revealed experimentally in colloidal suspensions of nanoparticles, the so-called nanofluids. The computed results show that the apparent thermal conductivity obtained via Fourier based relationships could indeed produce results showing substantial enhancement of the effective thermal conductivity calculated by means of the CV hyperbolic model. However, as the values of the times τ and τ_c are in general not well known, the interpretation of experimental data is difficult. On the other hand in Ref. [13] the authors show the results of numerical solutions of Eq. (7) for the significant case of a sample experiencing a sudden temperature change at the surface by heating using a pulsed laser beam. Finally, for a particular case of periodical modulated sample heating, Marín *et al* [18] demonstrated the very important result that for low modulation frequencies, when compared with $1/\tau$, the CV model leads to the conventional Fourier´s formalism. Inspired to a certain extent in the above mentioned work of Vadasz *et al.*, it has been recently suggested [19] the possibility to incorporate the so-called photothermal (PT) techniques [20], that use periodical heating to generate the measured signals, to the vast arsenal of methods used for thermal characterization, with the great advantage among these that the results of PT experiments can be interpreted using much simpler Fourier´s laws based models.

VI. CONCLUDING REMARKS

In conclusion, we hope that this brief article should be helpful to show people involved in the teaching and learning of heat transfer concepts that although it is well known that Fourier´s law in its initial form works, being compatible with daily experience, there are experimental situations where the temporal and/or spatial scales go down far from normal, making problematic its application. In these cases modifications such as the CV theory become necessary.

ACKNOWLEDGEMENTS

This work was partially financial supported by SIP-IPN and CONACyT. The support from COFAA-IPN through the SIBE Program is also acknowledged.

REFERENCES

[1] Fourier, J. B. J., *Analytical Theory of Heat,* (Encyclopedia Britannica, Inc., Chicago IL, 1952).

[2] Cattaneo, C., *Sulla conduzione de calore*, Atti Semin. Mat. Fis. Univ. Modena **3**, 83-101 (1948).

[3] Vernotte, P., *La véritable équation de la chaleur*, C. R. Acad. Sci., Paris **247**, 2103-2105 (1958).

[4] Özisik, M. N. and Vick, B., *Propagation and reflection of thermal waves in a finite médium*, Int. J. Heat and Mass Trans. **27**, 1845-1854 (1984).

[5] Joseph, D. D. and Preziosi, L., *Heat Waves*, Rev. Mod. Phys. **61**, 41-73 (1989); Joseph, D. D. and Preziosi, L., *Addendum to the paper "Heat Waves"*, Rev. Mod. Phys. **62**, 375-391 (1990).

[6] Band, W. and Meyer, L., *Second sound and the heat conductivity in helium II*, *Phys. Rev.* **73**, 226-229 (1948).

[7] Osborne, D. V., *Low Temperature Physics,* NBS Circular No 519 (US Govt Printing Office, Washington DC, 1950), p. 139.

[8] Tisza, L., *Sur la Supraconductibilit e thermique de l'helium II liquide et la statistique de Bose-. Einstein,* C. R. Acad. Sci., Paris **207**, 1035-1038 (1938).

[9] Landau, L., *Theory of the Superfluidity of Helium II*, J. Physique U.S.S.R. **5**, 71-90 (1941).

[10] Peshkov, V., *Second Sound in Helium II*, J. Physique U.S.S.R. **8**, 381-383 (1944).

[11] Tzou, D. Y., *The resonance phenomenon in thermal waves*, Int. J. Eng. Sc. **29**, 1167-1177 (1991).

[12] Tzou, D. Y., *Schock wave formation around a moving heat source in a solid with finite speed of heat propagation*, Int. J. Mass Heat Transfer. **32**, 1979-1987 (1989).

[13] Marín, E., Marin, J. and Hechavarría, R., *Hyperbolic heat diffusion in photothermal experiments with solids*, J. Phy. IV **125**, 365-368 (2005).

[14] Govender, M., Maartens, R. and Maharaj, S., *A causal model of radiating stellar collapse*, Phys. Lett. A, **283**, 71-79 (2001).

[15] Xiang-Qi, W., and Mujumdar, A. S., *Heat transfer characteristics in nanofluids: a review*, Int. J. of Th. Sci. **46**, 1-19 (2007); Cahill, D. G., Ford, W. K., Goodson, K. E., Mahan, G., Majumdar, D. A., Maris, H. J., Merlin, R. and Phillpot, S. R., *Nanoscale thermal transport,* J. Appl. Phys. **93**, 793-818, (2003).

[16] Greffet, J., *in Topics in Applied Physics*, edited by Voz S. (Springer, Paris 2007), pp. 1-13; Wolf E. L., *Nanophysics and Nanotechnology: An Introduction to Modern Concepts in Nanoscience,* (Wiley-VCH, Weinheim, 2004); E. Marín, *Characteristic dimensions for heat transfer,* Lat. Am. J. Phys. Educ. **4**, 56-60 (2010).

[17] Vadasz, J. J. *et al*., *Heat transfer enhancement in nano-fluids suspensions: Possible mechanisms and explanations*, Int. J. Heat Mass Transf. **48**, 2673-2683 (2005).

[18] Marín, E., Marín, J. and Díaz, P., *On the wave treatment of the conduction of heat in solids,* Eur. J. Phys. **23**, 523-532 (2002).

[19] Marín, E., Internet Electron., *On the role of photothermal techniques for the thermal characterization of nanofluids,* J. Nanoc. Moletrón. **5**, 1007-1014 (2007).

[20] Marin, E., (Ed.) *Thermal Wave Physics and Related Photothermal Techniques: Basic Principles and Recent Developments*, (Transworld Research, Kerala, 2009).