Lagrangian for the BMT equation

Alfonso Queijeiro Fontana
Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Edificio 9, U. P. Adolfo López Mateos, CP 07738, México D. F.

E-mail: aquei@esfm.ipn.

(Received 31 January 2011, accepted 30 May 2011)

Abstract
From the relativistic equation of motion for the spin, known as the BMT equation, we derive the corresponding Lagrangian function. The Euler-Lagrange equations are then used to recover the BMT equation, taking into account the non-commuting property of the spin coordinates. We also obtain the interaction Hamiltonian.

Keywords: Relativistic equation of motion, spin, Euler- Lagrange equations.

Resumen
A partir de la ecuación relativista de movimiento para el espín, conocida como la ecuación BMT, derivamos la función de Lagrange correspondiente. Las ecuaciones de Euler-Lagrange se utilizan para recuperar la ecuación de BMT, teniendo en cuenta las propiedades de no conmutatividad de las coordenadas de espín. Se obtiene también la Hamiltoniana de interacción.

Palabras clave: Ecuación relativista de movimiento, espín, ecuaciones de Euler-Lagrange.

PACS: 01.55. + b, 03.50. De **ISSN 1870-9095**

INTRODUCTION

In an analysis based on classical mechanics, Bargmann, Michel and Telegdi [1] derived in 1959, the relativistic equation describing the spin motion of particles in uniform and constant electric and magnetic fields [2]:

$$m\frac{d}{d\tau}S^{\alpha}=\frac{e}{mc}\left[\frac{g}{2}F^{\alpha\beta}S_{\beta}+\frac{1}{c^{2}}\left(\frac{g}{2}-1\right)U^{\alpha}\left(S_{\lambda}F^{\lambda\mu}U_{\mu}\right)\right], \quad (1)$$

here m and e is the mass and charge of the particle, S^{α} denotes the components of the spin 4-vector in some inertial reference frame, U^{α} is the particle´s 4-velocity, $F^{\alpha\beta}$ is Maxwell electromagnetic field tensor, and g and c are the Landé factor and the speed of light. Eq. (1) is the relativistic version of the equation of motion for the spin in its rest frame

$$\frac{d}{dt'}\vec{S}=\frac{eg}{2mc}\vec{S}\times\vec{B}'. \quad (2)$$

Here primes denote quantities defined in the rest frame.

The second summand in the right hand side of (1) is the anomaly of spin-1/2 particles, and is consequence of radiative corrections to the electromagnetic vertex. Shortly after the publication of Bargmann, Michel and Telegdi, Eq. (1) was named the BMT equation, and attempts were made to derive it from the Dirac equation in the limit of zero Planck constant [3]. The gyromagnetic ratio of a particle is the ratio of its magnetic moment to its intrinsic angular moment. For an elementary particle like the electron, the value $g=2(1+\frac{\alpha}{2\pi}+\cdots)$ is obtained in Quantum Electrodynamics, where α is the fine-structure constant, and the small correction to the result $g=2$ comes from radiative corrections.

On the other hand, spin is an intrinsic degree of freedom, i. e. it is an internal coordinate necessary to describe a physical state. Then, in a classical description the coordinate space has to be enlarged to include the spin degree. An important property of spin is that it is a non-commutative variable.

The purpose of this short note is to deduce the Lagrangian function for a charged particle with spin in an external electromagnetic field, starting from the equation of motion (1), for the case of $g=2$. We use the same method as in [4], where the Hilbert Lagrangian is derived from Einstein´s equation.

The starting point is Eq. (1) with $g=2$, written as

$$\dot{S}^{\mu}-\alpha F^{\mu\nu}S_{\nu}=0, \quad (3)$$

with $\alpha={}^{e}/_{m^{2}c}$. Multiplying (3) by δS_{μ} and integrating over $d\tau$ we obtain

$$0=\int d\tau\left(\dot{S}^{\mu}-\alpha F^{\mu\nu}S_{\nu}\right)\delta S_{\mu}. \quad (4)$$

Here δS_μ is a variation of the spin coordinate and is assumed it vanishes on the boundary points in the varied trajectory. Now, we rewrite the term $\dot{S}^\mu \delta S_\mu$ as

$$\dot{S}^\mu \delta S_\mu = \frac{d}{d\tau}\left(S^\mu \delta S_\mu\right) - S^\mu \frac{d}{d\tau}\delta S_\mu. \tag{5}$$

Then,

$$0 = \int d\tau \left[\frac{d}{d\tau}\left(S^\mu \delta S_\mu\right) - S^\mu \frac{d}{d\tau}\delta S_\mu - \alpha F^{\mu\nu} S_\nu \delta S_\mu\right]. \tag{6}$$

The first term in square brackets vanishes after integration and evaluation in the boundary point. We interchange the time derivative and the variation, in the second term, arriving to

$$0 = -\int d\tau \left[S^\mu \delta \dot{S_\mu} + \alpha F^{\mu\nu} S_\nu \delta S_\mu\right]. \tag{7}$$

The next step is to judiciously add and subtract terms leading to

$$\begin{aligned} 0 = -\int d\tau\, [\delta(S^\mu \dot{S}_\mu) - \dot{(\delta S^\mu)}\dot{S}_\mu + \delta(\alpha F^{\mu\nu} S_\nu S_\mu) \\ -\delta(\alpha F^{\mu\nu} S_\nu) S_\mu]. \end{aligned} \tag{8}$$

This can be grouped as

$$\begin{aligned} 0 = -\int d\tau\, \delta(S^\mu \dot{S}_\mu + \alpha F^{\mu\nu} S_\nu S_\mu) \\ + \int d\tau \left[(\delta S_\mu)\dot{S}^\mu + \alpha F^{\mu\nu} (\delta S_\nu) S_\mu\right. . \end{aligned} \tag{9}$$

Notice that term $\alpha F^{\mu\nu} S_\nu S_\mu$ does not vanish, since $S_\nu S_\mu = -S_\mu S_\nu$, and $F^{\mu\nu}$ is an anti-symmetric tensor. This fact is used to write

$$\boldsymbol{0 = -\int d\tau\, \delta L + \int d\tau \left(\delta S_\mu\right)\left(\dot{S}^\mu - \alpha F^{\mu\nu} S_\nu\right)}. \tag{10}$$

The last term is zero because the equation of motion itself, and the Lagrangian function L is identified with

$$L(S^\mu, \dot{S}^\mu) = S_\mu\left(\dot{S}^\mu - \alpha F^{\mu\nu} S_\nu\right), \tag{11}$$

a function identically zero but still allows one to derive the equations of motion [5].

To show that the equation of motion follows from (11), we use Lagrange equations

$$\frac{d}{d\tau}\frac{\partial}{\partial \dot{S_\lambda}} L - \frac{\partial}{\partial S_\lambda} L = 0. \tag{12}$$

From (11) we compute the different derivatives:

$$\frac{\partial}{\partial \dot{S_\lambda}} L = -S^\lambda, \tag{13a}$$

$$\frac{\partial}{\partial S_\lambda} L = \dot{S}^\lambda - 2\alpha F^{\lambda\nu} S_\nu. \tag{13b}$$

The minus sign in (13a) is a consequence of the anti-commuting character of the spin variables. That is, to operate $\partial / \partial \dot{S}_\lambda$ over L we must jump the factor S^μ, giving a minus sign [6]. Besides,

$$\frac{d}{d\tau}\frac{\partial}{\partial \dot{S}_\lambda} L = -\dot{S}^\lambda. \tag{14}$$

A substitution of (13) and (14) in (12) gives

$$-2(\dot{S}^\lambda - \alpha F^{\lambda\nu} S_\nu) = 0.$$

This is Eq. (3), except for a factor.

The non-relativistic limit of (3).

$$\dot{S}_i - \alpha \epsilon_{ijk} S_j B_k = 0, \tag{15}$$

and of (11)

$$L\left(S_i, \dot{S}_i\right) = S_i\left(\dot{S}_i - \alpha \epsilon_{ijk} S_j B_k\right), \tag{16}$$

may be obtained noticing that, in the system of reference of the particle, $S_\mu = (0, S_i)$ and the only non-zero components of the Maxwell tensor are the magnetic field components $F_{ij} = \epsilon_{ijk} B_k$.

We can construct the Hamiltonian function from the definition

$$H = \dot{S}^\mu \frac{\partial L}{\partial \dot{S_\lambda}} - L,$$

to obtain

$$\begin{aligned} H &= \dot{S}^\mu\left(-S_\mu\right) - S_\mu\left(\dot{S}^\mu - \alpha F^{\mu\nu} S_\nu\right) \\ &= \left(-S_\mu\right)\left(-\dot{S}^\mu\right) - S_\mu\left(\dot{S}^\mu - \alpha F^{\mu\nu} S_\nu\right) \\ &= \alpha F^{\mu\nu} S_\mu S_\nu. \end{aligned} \tag{17}$$

Again, we have used the anti-commuting property of the spin coordinates. In particle's system of reference (17) reduces to

$$H = \alpha \vec{B} \cdot \vec{S}. \tag{18}$$

This is the expression for the Hamiltonian of interaction between particle's magnetic moment and the external magnetic field.

ACKNOWLEDGEMENTS

This work was supported by COFAA-IPN and SIP-IPN.

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