The jerk vector in projectile motion

A. Tan and M. E. Edwards
Department of Physics, Alabama A & M University, Normal, Alabama 35762, U.S.A.

E-mail: arjun.tan@aamu.edu

(Received 12 February 2011, accepted 29 June 2011)

Abstract

The existence of the jerk vector is investigated in projectile motion under gravity. The jerk vector is zero in the absence of air resistance, but comes into life when velocity-dependent air drag is present. The jerk vector is calculated when the air resistance is proportional to the velocity. It is found that the jerk vector maintains a constant sense in the upward and forward direction, with its magnitude attenuated exponentially as a function of time.

Keywords: Jerk vector, Projectile motion, Air resistance.

Resumen

La existencia del vector reflejo es investigada en el movimiento de proyectiles en virtud de la gravedad. El vector reflejo es cero en ausencia de la resistencia del aire, pero vuelve a la vida cuando la resistencia del aire dependiente de la velocidad está presente. El vector reflejo es calculado cuando la resistencia del aire es proporcional a la velocidad. Se ha encontrado que el vector reflejo mantiene una sensación constante en la dirección hacia arriba y hacia delante, con su magnitud atenuada exponencialmente en función del tiempo.

Palabras clave: Vector reflejo, Movimiento de proyectiles, Resistencia del aire.

PACS: 45.50.-j, 45.50.Dd, 45.40.Gj, 45.40.Aa **ISSN 1870-9095**

I. INTRODUCTION

The vast majority of physical laws are represented by second order differential equations. This is because the first and second derivatives of the pertinent physical quantities give, respectively, the slope and curvature of these quantities, which are sufficient for the solution of the problem at hand. In kinematics, for example, the velocity and acceleration are defined as the first and second derivatives, respectively, of the position vector with respect to time. The third derivative is normally overlooked in the vast majority of the textbooks in undergraduate and graduate physics curricula. Yet the existence of this derivative has been known for a long time. Formerly known as the *second acceleration*, this derivative is now commonly referred to as the *jerk* [1]. The jerk vector has been studied in simple harmonic motion [2], uniform circular motion [3] and Keplerian motion [2, 4]. In this paper, we investigate the existence of the jerk vector in projectile motion.

II. PROJECTILE MOTION WITHOUT AIR RESISTANCE

The problem of projectile motion under gravity in the absence of air resistance forms one of the basis topics in elementary physics courses [5]. In the standard treatment, a projectile of mass m is projected with an initial velocity $\vec{v}_0$ making an angle α with the horizontal plane. In a Cartesian coordinate system x-y with the launch point at the origin, the equations of motion are given by

$$m\frac{dv_x}{dt} = 0 , \tag{1}$$

and

$$m\frac{dv_y}{dt} = -mg . \tag{2}$$

The initial conditions at time $t = 0$ are given by $v_x = v_0 \cos\alpha$; $v_y = v_0 \sin\alpha$; $x = 0$; $y = 0$.

Eqs. (1) and (2) are uncoupled equations which can be integrated separately. Integrating them twice with respect to t and applying the initial conditions, we obtain

$$v_x = v_0 \cos\alpha , \tag{3}$$

$$v_y = v_0 \sin\alpha - gt , \tag{4}$$

$$x = v_0 \cos\alpha t , \tag{5}$$

and

$$y = v_0 \sin\alpha t - \frac{1}{2} g t^2 . \tag{6}$$

Eqs. (3) – (6) furnish the position and velocity vectors of the projectile as functions of time:

$$\vec{r} = \vec{v}_0 - \frac{1}{g} g t^2 \hat{y}\,, \qquad (7)$$

and

$$\vec{a} = -g\hat{y}\,. \qquad (8)$$

A further differentiation of Eq. (8) supplies the jerk vector:

$$\vec{j} = \vec{0}\,. \qquad (9)$$

Thus, for projectile motion under gravity in the absence of air resistance, the jerk vector is zero. This is to be expected since the force (and hence the acceleration) of the projectile is a constant vector in the downward direction. We shall next show that the jerk vector will not be a null vector in projectile motion if velocity-dependent drag forces are present.

III. PROJECTILE MOTION WITH LINEAR AIR RESISTANCE

Two types of air resistance are commonly found in the literature: (1) proportional to the velocity (*linear drag*); and (2) proportional to the square of the velocity (*quadratic drag*) [6]. The general effects of both types of drag forces on the projectile motion are similar. However, exact solutions of the problem in time are found only in the linear case [7]. Hence, we shall restrict ourselves to the linear drag force in this paper. Also, it is advantageous to express the drag force as $-km\vec{v}$, where k is the drag coefficient per unit mass [6]. The equations of motion are then written as

$$m\frac{dv_x}{dt} = -kmv_x\,, \qquad (10)$$

and

$$m\frac{dv_y}{dt} = -mg - kmv_y\,. \qquad (11)$$

Eqs. (10) and (11) are still uncoupled and thus can be integrated separately. We assume the same initial conditions as before.

The horizontal motion is solved by separating the variables v_x and t and integrating both sides from the initial conditions to give

$$v_x = v_0 \cos\alpha e^{-kt}\,. \qquad (12)$$

Thus, from Eq. (10), one gets

$$a_x = \frac{dv_x}{dt} = -kv_0 \cos\alpha e^{-kt}\,, \qquad (13)$$

whence

$$j_x = \frac{da_x}{dt} = k^2 v_0 \cos\alpha e^{-kt}\,. \qquad (14)$$

The vertical motion can likewise be solved, giving

$$v_y = -\frac{g}{k} + \frac{kv_0 \sin\alpha + g}{k} e^{-kt}\,. \qquad (15)$$

From Eq. (11),

$$a_y = \frac{dv_y}{dt} = -(kv_0 \sin\alpha + g)e^{-kt}\,, \qquad (16)$$

whence

$$j_y = \frac{da_y}{dt} = k(kv_0 \sin\alpha + g)e^{-kt}\,. \qquad (17)$$

The velocity, acceleration and jerk vectors of the projectile can be constructed from Eqs. (12 - 17):

$$\vec{v} = v_x\hat{x} + v_y\hat{y} = e^{-kt}\vec{v}_0 - \frac{g}{k}\left(1 - e^{-kt}\right)\hat{y}\,, \qquad (18)$$

$$\vec{a} = a_x\hat{x} + a_y\hat{y} = -ke^{-kt}\vec{v}_0 - ge^{-kt}\hat{y}\,, \qquad (19)$$

and

$$\vec{j} = j_x\hat{x} + j_y\hat{y} = k^2 e^{-kt}\vec{v}_0 - kge^{-kt}\hat{y}\,. \qquad (20)$$

Eqs. (19) and (20) can be rewritten as

$$\vec{a} = -(k\vec{v}_0 + g\hat{y})e^{-kt}\,, \qquad (21)$$

and

$$\vec{j} = k(k\vec{v}_0 + g\hat{y})e^{-kt}\,. \qquad (22)$$

By inspection, $k\vec{v}_0 + g\hat{y}$ is a constant vector whose sense is between the forward and upward directions. Also, e^{-kt} is an attenuation factor which diminishes exponentially in time. Consequently, the jerk vector maintains a constant direction (between the forward and upward directions) while its magnitude diminished exponentially in time. The constant slope angle θ of this vector is given by

$$\theta = \tan^{-1}\frac{kv_0 \sin\alpha + g}{kv_0 \cos\alpha} = \tan^{-1}\left(\tan\alpha + \frac{g}{kv_0}\sec\alpha\right). \qquad (23)$$

Further, from Eqs. (21) and (22), we have

$$\vec{j} = -k\vec{a}\,, \qquad (24)$$

which indicates that the jerk vector or second acceleration is anti-parallel to the acceleration vector (which is between the downward and backward directions). It may be mentioned that in uniform circular motion, the jerk vector is anti-parallel to the velocity vector and perpendicular to the acceleration vector [3].

It is instructive to calculate the times of ascent (τ_1) and descent (τ_2) of the projectile under gravity subject to linear drag force. At the apex of the trajectory, $vy = 0$. Thus, from Eq. (15):

$$\tau_1 = -\frac{1}{k} \ln \frac{g}{kv_0 \sin\alpha + g} . \quad (25)$$

To obtain the time of flight τ (= τ_1 + τ_2), we integrate Eq. (15) with respect to t and apply the initial conditions getting

$$y = -\frac{gt}{k} + \frac{kv_0 \sin\alpha + g}{k^2}\left(1 - e^{-kt}\right). \quad (26)$$

The time of flight is obtained when $y = 0$. One arrives at a transcendental equation

$$f_1(t) = f_2(t) , \quad (27)$$

with

$$f_1(t) = t , \quad (28)$$

and

$$f_2(t) = \frac{kv_0 \sin\alpha + g}{gk}\left(1 - e^{-kt}\right). \quad (29)$$

Eq. (27) can be solved by the graphical method giving τ. In Fig.1, the intersection of the curves $f_1(t)$ and $f_2(t)$ furnishes the value of τ. The time of descent τ_2 is then obtained by subtracting τ_1 from τ.

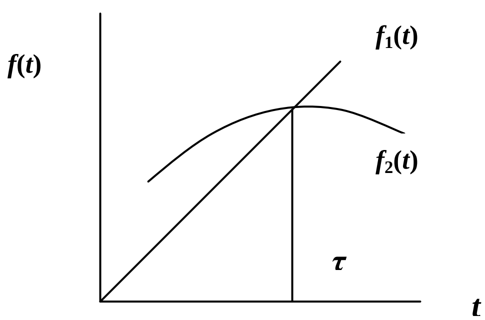

FIGURE 1. Graphical solution of Eq. (27). The intersection of $f_1(t)$ and $f_2(t)$ furnishes the value of τ.

IV. EXAMPLE OF JERK VECTOR IN PROJECTILE MOTION

We now illustrate the treatment of the last section with an example. Let the projection speed v_0 = 600m/s; the launch angle $\alpha = 60^o$; and the linear drag coefficient per unit mass k = .01/s. Assume g = 9.8m/s^2. The time of ascent as given by Eq. (25) is τ_1 = 42.5s. The time of flight as obtained from the graphical solution of Eq. (27) is τ = 92.1s. The time of descent follows from the above: τ_2 = 49.6s. We have $\tau_2 > \tau_1$ as expected.

The initial magnitude of the jerk vector is obtained by setting $t = 0$ in Eqs. (14) and (17) and applying Pythagoras' theorem:

$$j_0 = k\sqrt{k^2 v_0^{\,2} + 2kv_0 g \sin\alpha + g^2} , \quad (30)$$

giving j_0 = .153m/s^3. The slope angle of the jerk vector as given by Eq. (23) is $\theta = 76.4^o$. Fig. 2 shows the trajectory of the projectile. Also displayed in the figure are the jerk vectors at intervals of 10s. As noted earlier, the vector maintains a fixed direction while its amplitude diminished steadily in time.

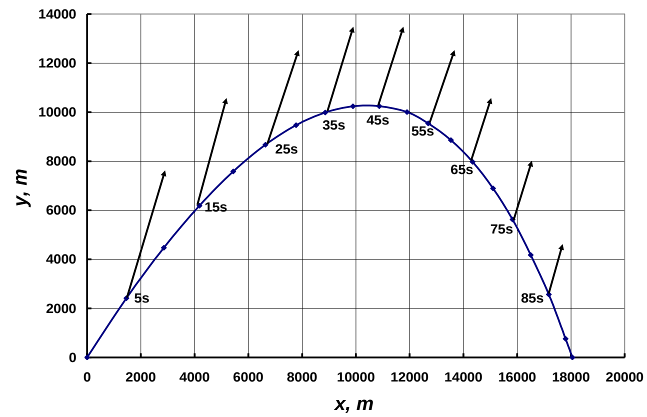

FIGURE 2. The jerk vector in projectile motion with launch speed v_0 = 600m/s; launch angle $\alpha = 60^o$; and drag coefficient k = .01/s.

V. DISCUSSION

The existence of the jerk vector in projectile has been investigated. The absence of air resistance ensures the absence of the jerk vector, since the constant gravitational force cannot produce jerk. It is air resistance which is responsible for creating jerk in projectile motion. The jerk is quite a common phenomenon, even though it is not a part of of the physics curricula. A variable force will always produce a variable acceleration, and hence jerk. The jek vector in projectile motion with air drag is only another example of this common but ignored phenomenon.

REFERENCES

[1] Campbell, N. R., *Physics: The Elements*, (Cambridge University Press, Cambridge, 1920), p. 42.
[2] Schot, S. H., *The time rate of change of acceleration*, Am. J. Phys. **46**, 1090-1094 (1978).
[3] Sandin, T. R., *The jerk*, Phys. Teach. **28**, 36-38 (1990).
[4] Tan, A., *Theory of Orbital motion*, (World Scientific, USA, 2008), pp. 64-69.

[5] Resnick, R., Halliday, D. and Krane, K. S., *Physics*, Vol. 1 (John Wiley, USA, 1992), pp. 57-58.
[6] Marion, J. B. and Thornton, S. T., *Classical Dynamics of particles and systems*, (Saunders College Publishing, USA, 1995), pp. 58-60.
[7] Tan, A., Frick, C. H. and Castillo, O., *The fly ball trajectory: An older approach revisited*, Am. J. Phys. **55**, 33-40 (1987).