Dynamical variables in brachistochrone problem

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Abstract

The historic Brachistochrone problem is widely discussed in the literature. However, the discussion is primarily limited to the shape of the curve along which a particle will descend, under gravity, and in the absence of friction, from a point to another not directly below it, in the shortest amount of time. This study examines the various dynamical variables associated with this motion, including the velocity, acceleration and jerk vectors, along with kinetic, potential and total energies, curvature and centripetal force, as the particle undertakes its journey. The quantities are expressed as functions of the angular parameter. The acceleration and jerk vectors are found to have constant magnitudes and to rotate counter-clockwise, with the former trailing the latter by 90°. The velocity and centripetal force vectors also rotate counter-clockwise, but with half the angular velocity, with the former trailing the latter by 90°. This study further examines how the dynamical variables are affected when kinetic friction is present.

Keywords: Brachistochrone Problem, Dynamical Variables, Jerk vector, Curvature.

Resumen

El problema histórico de la Braquistócrona es ampliamente discutido en la literatura. Sin embargo, la discusión se limita principalmente a la forma de la curva a lo largo de la cual una partícula descenderá, por gravedad, y en ausencia de fricción, de un punto a otro no directamente debajo de él, en el menor tiempo posible. Este estudio analiza las diferentes variables dinámicas asociadas con este movimiento, incluyendo los vectores de velocidad, aceleración y de jerk, junto con las energías cinética, potencial y energías totales, la curvatura y fuerza centrípeta, como la partícula lleva a cabo su trayectoria. Las cantidades son expresadas como funciones del parámetro angular. Los vectores de aceleración y de jerk son encontrados con magnitudes constantes y giran en sentido-antihorario, con la primera final de la segunda por 90°. La velocidad y los vectores de fuerza centrípeta también giran en sentido-antihorario, pero con la mitad de la velocidad angular, con la primera final de la segunda por 90°. Este estudio además examina cómo las variables dinámicas son afectadas cuando la fricción cinética está presente.

Palabras clave: Problema de la Braquistócrona, Variables Dinámicas, Vector Jerk, Curvatura.

I. INTRODUCTION

In 1696, Johann Bernoulli posed the Brachistochrone problem as a challenge to the other mathematicians of the day: To find the curve along which a particle will descend, under gravity, from a point to another not directly under it, in the shortest amount of time. The problem was correctly solved by his elder brother Jakob Bernoulli, as well as by Newton, Leibniz and L'Hospital, giving the segment of an inverted cycloid as the answer [1, 2, 3]. It is this historic problem which gave rise to the new branch of mathematics called the *Calculus of Variations*. Many textbooks have devoted pages to this famous problem, but invariably, the discussion ends abruptly upon finding the curve. In this paper, we examine the various dynamical variables associated with this motion, including the velocity, acceleration and jerk vectors, along with kinetic, potential and total energies, curvature and centripetal force, as the particle undertakes its journey.

Among the dynamical variables, we include the *jerk*, which is the derivative of the acceleration vector, or the third derivative of the position vector, with respect to time. The jerk vector has recently been studied in projectile motion [4] and motion of charged particles [5]. As by-products of the jerk, one obtains the *curvature* and *torsion* of the path. If the first three derivatives of the position vector in time, viz., the velocity, acceleration and the jerk vectors are $\vec{v}$, $\vec{a}$ and $\vec{j}$, respectively, then the curvature κ and torsion τ are given by [5]:

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}, \tag{1}$$

and

$$\tau = \frac{|\vec{v} \circ (\vec{a} \times \vec{j})|}{|\vec{v} \times \vec{a}|^2}. \tag{2}$$

The reciprocal of the curvature furnishes the radius of curvature, which in turn, gives the centripetal force.

II. THE BRACHISTOCHRONE PROBLEM

Consider the problem of a particle descending under gravity from a point at the origin $(0, 0)$ to another (x, y) in the *x-y* plane, not directly under the first (Fig. 1). Determine the path along which the particle will slide, without friction, in the shortest time. For the sake of later convenience, reckon *y* to be positive downwards. This is a conservative system, for which the total energy remains constant. Further, if the particle slides from the rest $E = T + V = 0$. In the usual notions,

$$\frac{1}{2}mv^2 - mgy = 0 . \tag{3}$$

The time of passage between the two points is:

$$\tau = \int dt = \int \frac{ds}{v} = \int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int \sqrt{\frac{1 + y'^2}{y}} dx , \tag{4}$$

with $y' = dy / dx$. For τ to be minimum, the integrand

$$f = f(y, y', x) = \sqrt{\frac{1 + y'^2}{y}} , \tag{5}$$

must satisfy the *Euler-Lagrange equation*:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 . \tag{6}$$

When *f* does not contain *x* explicitly, the Euler-Lagrange equation reduces to *Beltrami's Identity*:

$$f - y'\frac{\partial f}{\partial y'} = const. \tag{7}$$

Thus,

$$\frac{1}{\sqrt{y}\sqrt{1 + y'^2}} = const. \tag{8}$$

Squaring and rearranging:

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right] y = 2A , \tag{9}$$

where *A* is another constant. One can verify that the following equations constitute the solution to Eq. (9):

$$x = A(\theta - \sin \theta) , \tag{10}$$

and

$$y = A(1 - \cos \theta) . \tag{11}$$

Remembering that *y* is positive downwards, Eqs. (10) and (11) are recognized as the parametric equations of an inverted cycloid, *i.e.*, a curve traced out by a point on a circle of radius *A* rolling under the positive *x*-axis (Fig. 1).
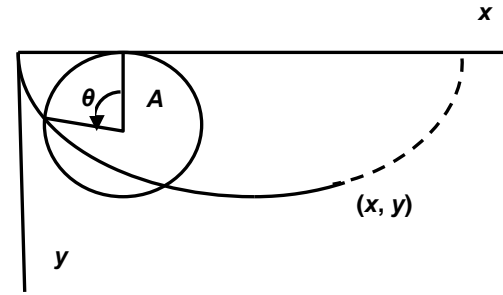

FIGURE 1. Path of descent of particle in minimum time.

III. DYNAMICAL VARIABLES IN THE BRACHISTOCHRONE PROBLEM

Starting from the position vector, the velocity, acceleration and jerk vectors can be calculated by successive differentiation of the position vector with respect to time. We have

$$\vec{r} = A(\theta - \sin \theta)\hat{x} + A(1 - \cos \theta)\hat{y} . \tag{12}$$

Letting $\theta = \omega t$, where ω is the angular velocity of the rolling circle, one obtains:

$$\vec{v} = A\omega(1 - \cos \theta)\hat{x} + A\omega \sin \theta \hat{y} , \tag{13}$$

$$\vec{a} = A\omega^2 \sin \theta \hat{x} + A\omega^2 \cos \theta \hat{y} , \tag{14}$$

and

$$\vec{j} = A\omega^3 \cos \theta \hat{x} - A\omega^3 \sin \theta \hat{y} . \tag{15}$$

The magnitudes of the above quantities are:

$$v = |\vec{v}| = \sqrt{2}A\omega\sqrt{1 - \cos \theta} , \tag{16}$$

$$a = |\vec{a}| = A\omega^2 , \tag{17}$$

and

$$j = |\vec{j}| = A\omega^3 . \tag{18}$$

Also,

$$v^2 = 2A^2\omega^2\left(1-\cos\theta\right), \tag{19}$$

and

$$v^3 = 2\sqrt{2}A^3\omega^3\left(1-\cos\theta\right)^{3/2}, \tag{20}$$

We have, further:

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A\omega(1-\cos\theta) & A\omega\sin\theta & 0 \\ A\omega^2\sin\theta & A\omega^2\cos\theta & 0 \end{vmatrix} \tag{21}$$

$$= -A^2\omega^3\left(1-\cos\theta\right)\hat{z}.$$

$$\vec{a}\times\vec{j} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A\omega^2\sin\theta & A\omega^2\cos\theta & 0 \\ A\omega^3\cos\theta & -A\omega^3\sin\theta & 0 \end{vmatrix} \tag{22}$$

$$= -A^2\omega^5\hat{z}.$$

$$\left|\vec{v}\times\vec{a}\right| = A^2\omega^3\left(1-\cos\theta\right), \tag{23}$$

and

$$\vec{v}\circ\left(\vec{a}\times\vec{j}\right) = 0. \tag{24}$$

Eqs. (1), (20) and (23) give the value of the curvature κ and that of its reciprocal (the radius of curvature) R:

$$\kappa = \frac{1}{2\sqrt{2}A\sqrt{1-\cos\theta}}, \tag{25}$$

and

$$R = \frac{1}{\kappa} = 2\sqrt{2}A\sqrt{1-\cos\theta}. \tag{26}$$

Eqs. (19) and (26) furnish the value of the centripetal force:

$$F = \frac{mv^2}{R} = \frac{1}{\sqrt{2}}mA\omega^2\sqrt{1-\cos\theta}. \tag{27}$$

Eqs. (2) and (24) indicate that the torsion of the path is zero. This is to be expected as the motion takes place in a vertical plane, and the torsion is rate of turning of the tangent vector out of the plane.

The kinetic energy of the particle is obtained from Eq. (19):

$$T = \frac{1}{2}mv^2 = mA^2\omega^2\left(1-\cos\theta\right). \tag{28}$$

Likewise, the potential energy is [from Eq. (12)]:

$$V = -mgy = -mgA\left(1-\cos\theta\right). \tag{29}$$

The conservation of total energy yields the value of the angular velocity of the rolling circle: $\omega = \sqrt{A/g}$. In the above equations, the dynamical quantities are conveniently expressed as functions of the angle θ. If the motion spans the entire cycloid, then θ runs from 0 to 2π, and the duration of the motion is $2\pi/\omega$. The horizontal length of the cycloid is $2\pi A$ and depth of the trajectory, midways, at the lowest point ($\theta = \pi$), is $y = 2A$. The angular dependences of the dynamical variables are shown in Table I.

TABLE I. Angular dependence of Dynamical Variables.

Dynamical variable	Formula	θ-dependence
Abscissa	Eq. (10)	$\theta - \sin\theta$
Ordinate	Eq. (11)	$1-\cos\theta$
x-component of velocity	$v_x = \dot{x}$	$1-\cos\theta$
y-component of velocity	$v_y = \dot{y}$	$\sin\theta$
x-component of acceleration	$a_x = \ddot{x}$	$\sin\theta$
y-component of acceleration	$a_y = \ddot{y}$	$\cos\theta$
x-component of jerk	$j_x = \dddot{x}$	$\cos\theta$
y-component of jerk	$j_y = \dddot{y}$	$-\sin\theta$
Speed	$v = \left|\vec{v}\right|$	$\sqrt{1-\cos\theta}$
Magnitude of acceleration	$a = \left|\vec{a}\right|$	constant
Magnitude of jerk	$j = \left|\vec{j}\right|$	constant
Momentum	$\vec{p} = m\vec{v}$	$\sqrt{1-\cos\theta}$
Kinetic energy	Eq. (28)	$1-\cos\theta$
Potential energy	Eq. (29)	$\cos\theta - 1$
Total energy	$E = T+V$	constant 0
Curvature	Eq. (25)	$\frac{1}{\sqrt{1-\cos\theta}}$
Torsion	Eq. (2)	0
Radius of curvature	Eq. (26)	$\sqrt{1-\cos\theta}$
Centripetal force	Eq. (27)	$\sqrt{1-\cos\theta}$

Eqs. (17) and (18) indicate that the acceleration and jerk vectors possess constant magnitudes throughout the motion. The speed of the particle, on the other hand, varies, being zero at the onset of the motion, reaching a maximum at the middle of the cycloid, and becoming zero again at the terminus. The angles the velocity, acceleration and jerk vectors make with the positive x-axis can conveniently be expressed in terms of the angle θ. Let the above angles be denoted by α, β and γ, respectively. Then, we have:

$$\alpha = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{\sin\theta}{1-\cos\theta} = \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \frac{\pi}{2}-\frac{\theta}{2}, \tag{30}$$

$$\beta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\left(\cot\theta\right) = \frac{\pi}{2}-\theta, \tag{31}$$

and

$$\gamma = \tan^{-1}\frac{j_y}{j_x} = \tan^{-1}\left(-\tan\theta\right) = -\theta. \qquad (32)$$

Since y is positive downwards, the angles are reckoned positive if clockwise. Thus the acceleration and jerk vectors rotate counter-clockwise with θ, with the former trailing the latter by 90°. The velocity vector, too, rotates counter-clockwise, but with half the angular speed. Also, since the centripetal force is always perpendicular to the velocity, it too, rotates counter-clockwise with the same angular speed, leading the velocity vector by 90°. The velocity, acceleration and jerk vectors are depicted at intervals of 90°s in Fig. 2.
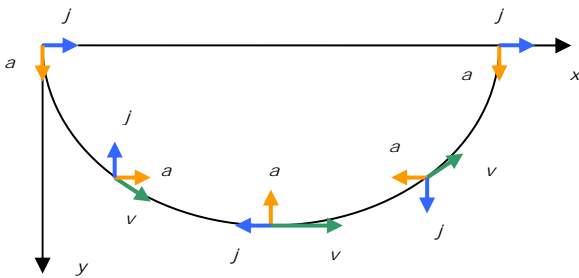

FIGURE 2. Velocity, acceleration and jerk vectors at intervals of 90°.

IV. BRACHISTOCHRONE PROBLEM WITH KINETIC FRICTION

When kinetic friction is included, the Brachistochrone problem becomes much more formidable. A closed form of solution was obtained, but the solution is quite intractable, and the equation of motion could not be integrated to give the velocity as a function of position [6]. An approximate solution was found, with neglect of the centripetal force [7], which was easy to work with. If μ is the coefficient of kinetic friction, we have, instead of Eqs. (10) and (11) (vide [7]):

$$x = A\left[\left(\theta - \sin\theta\right) + \mu\left(1 - \cos\theta\right)\right], \qquad (33)$$

and

$$y = A\left[\left(1 - \cos\theta\right) + \mu\left(\theta + \sin\theta\right)\right]. \qquad (34)$$

Fig. 3 displays the trajectories given by Eqs. (33) and (34) for three values of μ equal to 0.1, 0.2 and 0.3.
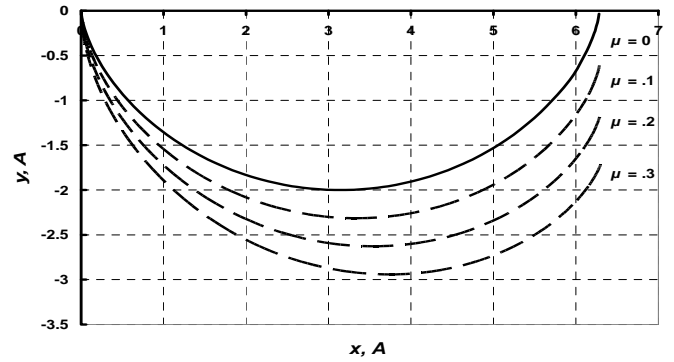

FIGURE 3. Brachistochrone trajectories with various coefficients of kinetic friction.

The dynamical variables are conveniently calculated using Eqs. (33) and (34) following the usual procedure. We have

$$\vec{v} = A\omega\left[\left(-\cos\theta + \mu\sin\theta\right)\hat{x} + \left(\sin\theta + \mu + \mu\cos\theta\right)\hat{y}\right]. \quad (35)$$

$$\vec{a} = A\omega^2\left[\left(\sin\theta + \mu\cos\theta\right)\hat{x} + \left(\cos\theta - \mu\sin\theta\right)\hat{y}\right]. \qquad (36)$$

$$\vec{j} = A\omega^3\left[\left(\cos\theta - \mu\sin\theta\right)\hat{x} - \left(\sin\theta + \mu\cos\theta\right)\hat{y}\right]. \qquad (37)$$

$$\vec{v}\times\vec{a} = A^2\omega^3\left[\left(1-\mu^2\right)\cos\theta - \left(1+\mu^2\right) - 2\mu\sin\theta\right]\hat{z}. \quad (38)$$

$$\left|\vec{v}\times\vec{a}\right| = A^2\omega^3\left[\left(1+\mu^2\right) - \left(1-\mu^2\right)\cos\theta + 2\mu\sin\theta\right]. \qquad (39)$$

$$v^2 = 2A^2\omega^2\left[\left(1+\mu^2\right) - \left(1-\mu^2\right)\cos\theta + 2\mu\sin\theta\right]. \qquad (40)$$

$$v^3 = 2\sqrt{2}A^3\omega^3\left[\left(1+\mu^2\right) - \left(1-\mu^2\right)\cos\theta + 2\mu\sin\theta\right]^{3/2}. \quad (41)$$

$$\kappa = \frac{1}{2\sqrt{2}A\sqrt{\left(1+\mu^2\right) - \left(1-\mu^2\right)\cos\theta + 2\mu\sin\theta}}. \qquad (42)$$

$$a = A\omega^2\sqrt{1+\mu^2}. \qquad (43)$$

$$j = A\omega^3\sqrt{1+\mu^2}. \qquad (44)$$

$$R = \frac{1}{\kappa} = 2\sqrt{2}A\sqrt{\left(1+\mu^2\right) - \left(1-\mu^2\right)\cos\theta + 2\mu\sin\theta}, \quad (45)$$

and

$$F = \frac{mv^2}{R} = \frac{1}{\sqrt{2}}mA\omega^2\sqrt{\left(1+\mu^2\right) - \left(1-\mu^2\right)\cos\theta + 2\mu\sin\theta}. \qquad (46)$$

The angular dependences of these variables are shown in Table II. All quantities are affected by friction but the magnitudes of the acceleration and jerk vectors remain constants. It is easy to verify that the entries of Table II reduce to those of Table I in the absence of friction.

TABLE II. Angular dependences of Dynamical Variables with kinetic friction.

Dynamical variable	θ-dependence
Abscissa	$(\theta - \sin\theta) + \mu(1 - \cos\theta)$
Ordinate	$(1 - \cos\theta) + \mu(\theta + \sin\theta)$
x-component of velocity	$(1 - \cos\theta) + \mu\sin\theta$
y-component of velocity	$\sin\theta + \mu(1 + \cos\theta)$
x-component of acceleration	$\sin\theta + \mu\cos\theta$
y-component of acceleration	$\cos\theta - \mu\sin\theta$
x-component of jerk	$\cos\theta - \mu\sin\theta$
y-component of jerk	$-(\sin\theta + \mu\cos\theta)$
Speed	$\sqrt{(1+\mu)^2 - (1-\mu)^2\cos\theta + 2\mu\sin\theta}$
Magnitude of acceleration	constant
Magnitude of jerk	constant
Momentum	$\sqrt{(1+\mu)^2 - (1-\mu)^2\cos\theta + 2\mu\sin\theta}$
Kinetic energy	$(1+\mu)^2 - (1-\mu)^2\cos\theta + 2\mu\sin\theta$
Potential energy	$(\cos\theta - 1) - \mu(\theta + \sin\theta)$
Curvature	$\dfrac{1}{\sqrt{(1+\mu)^2 - (1-\mu)^2\cos\theta + 2\mu\sin\theta}}$
Torsion	0
Radius of curvature	$\sqrt{(1+\mu)^2 - (1-\mu)^2\cos\theta + 2\mu\sin\theta}$
Centripetal force	$\sqrt{(1+\mu)^2 - (1-\mu)^2\cos\theta + 2\mu\sin\theta}$

V. CONCLUSIONS

The historic Brachistochrone problem has been associated with the greatest mathematical minds of that time, including the Bernoulli brothers, Newton, Leibniz and L'Hospital, all of whom can be regarded as founders or co-founders of Calculus. This problem also gave birth to the new branch of mathematics called the Calculus of Variations. Quite surprisingly, the discussion of this problem in the literature is invariably related to finding the shape of the trajectory, and the dynamical aspects of the problem are completely ignored. It is hoped that this study fills an important void in the dynamical aspects of this fascinating problem.

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