Some considerations on quantities of dimension one in oscillatory motions

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Abstract
The habit of not writing a multiplicative factor of dimension one leads to equations, which, though dimensionally correct, can pose some problems to most students as for the analysis of the units of measurement of the written physical quantities. The proposed analysis of the derivation of the equations for a simple oscillator and a forced oscillator with damping shows how to handle with the units of measurement. Similar considerations can be done for the formula of error propagation for oscillating functions.

Keywords: Units of measurement, Radian, Oscillatory motion.

Resumen
El hábito de no escribir un factor multiplicativo de una dimensión conduce a las ecuaciones, las cuales, aunque de dimensiones correctas, pueden plantear algunos problemas a la mayoría de los estudiantes como para el análisis de las unidades de medida de las cantidades físicas escritas. El análisis propuesto de la derivación de las ecuaciones para un oscilador simple y un oscilador forzado con amortiguamiento muestra cómo controlar a las unidades de medida. Se pueden hacer consideraciones similares para la fórmula de propagación de errores para funciones oscilantes.

Palabras clave: Unidades de medición, Radián, Movimiento oscilatorio.

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I. INTRODUCTION

The study of oscillatory motions in the most common physics textbooks used in undergraduate courses (see, *e.g*., [1, 2, 3, 4, 5]) is based on definitions and laws described in previous chapters (*e.g.*: Cinematic, Newton's laws) and on theorems and their applications studied in calculus courses. Dimensional analysis and the use of the S.I. units of measurement are commonly explained in one of the introductory chapters. In later chapters, the students' attention is usually drawn only to the units of measurement of the newly introduced physical quantities, whereas very few words if any are spent about the dimensional analysis of physical equations. In the study of the oscillatory motions, the radian is used as the S.I. unit of measurement of angle, which is a physical quantity of dimension one. Because of the habit of not explicitly writing a multiplicative factor of 1rad, most students experience problems in completing the analysis of the units of measurements of the oscillatory motion equations. A careful analysis of the step-by-step derivation of the equations of motion for any case of simple oscillators shows how to face this kind of problems. The case of a forced oscillator with damping can be handled with in a similar way.

II. A PROBLEM WITH THE UNIT OF MEASUREMENT OF THE ANGULAR FREQUENCY

The frequency ν of an oscillatory motion is defined as the number of oscillation cycles per time. Its S.I. unit of measurement is therefore the cycle per second (usually called hertz). The period of oscillation T is defined as the inverse of the frequency. It is the duration of one complete oscillation cycle. Its S.I. unit of measurement is the second (per cycle). The angular frequency ω can be defined as a function of the frequency (or of the period) as $\omega = 2\pi\nu$, where the proportionality factor is 2π radian per cycle. Its S.I. unit of measurement is the radian per second. If we make use of the mathematical tool of considering an oscillatory motion as the projection of a uniform circular motion on one axis, then the angular frequency can be considered as the angular speed of the uniform circular motion. The S.I. unit of measurement of angular speed is also the radian per second.

The angular frequency can be calculated as a function of some opportunely chosen physical properties of the particular oscillator under consideration. Let us consider a mass-spring system as an example of a simple oscillator. In this case $\omega^2 = k/m$, where k is the spring elastic constant (measured in newtons per meter), and m is the mass attached to the spring (measured in kilograms). Since the newton is defined as 1N = 1kg m s^{-2}, when we try to determine the S.I. unit of measurement of the angular frequency from $\omega^2 = k/m$ we find that it is the reciprocal second, and not the radian per second.

At this point, most students are not able to understand why the radian disappears and where it is hidden. The BIPM suggests the habit to generally omit the symbol "rad" when it concerns the unit one with physical quantities of dimension one. This explanation is not considered a satisfactory one by most students. To help them face this problem, we suggest to see in deeper detail how a relation such as $\omega^2 = k/m$ is obtained.

Similar considerations can be done with simple oscillators of other kinds: the simple pendulum $\omega^2 = g/l$, the physical pendulum $\omega^2 = mgr/J$, the LC-circuit $\omega^2 = 1/LC$, etc., which all give the reciprocal second as unit of measurement of the angular frequency.

III. THE SIMPLE OSCILLATOR

Newton's law $F = ma$ for the mass-spring system can be written, using Hooke's law for the elastic force, as $ma = -kx$, where m is the mass (measured in kilograms), a is the acceleration (in meters per second squared), k is the elasticity constant (in newtons per meter), and x is the position (in meters).

The acceleration a is, by definition, the second time-derivative of the position x. We can therefore write Newton's law for the mass-spring system as:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x\,. \tag{1}$$

We require that the function $x = x(t)$ is limited in space. The general solution of Eq. (1) is a linear combination of sine and cosine functions. Without loss of generality, let us consider, for simplicity, the following solution of (1):

$$x(t) = A\cos(\omega t + \varphi). \tag{2}$$

The amplitude A has the same unit of measurement of x; in our example, it is therefore measured in meters. The angular frequency ω is measured in radians per second. The phase φ is measured in radians.

The use of the radian as the unit of measurement of the cosine function argument is due to the fact that the derivative formula

$$\frac{d\cos}{d\theta}(\theta) = -\sin(\theta)\,, \tag{3}$$

is valid only if θ is measured in radians, and not in other non-S.I. units of measurement of angle such as the sexagesimal degree. The corresponding formulae for other trigonometric functions can be found in any calculus textbook.

Once we know the value of θ in radians, we can calculate $\cos(\theta)$ and $\sin(\theta)$, which have dimension one and have no unit of measurement. At this point, both $\cos(\theta)$ and its derivative $-\sin(\theta)$ have no unit of measurement. At the same time, both sides of (3) must be measured with the same unit of measurement, so that it is better for our purposes to explicitly consider the unit of measurement of $-\sin(\theta)$ to be the reciprocal radian.

In order to make the students better understand this point, we might consider writing (3) using step-by-step the definition of derivative itself as the limit of an incremental ratio:

$$\frac{d\cos}{d\theta}(\theta) = \lim_{h\to 0}\frac{\cos(\theta+h)-\cos(\theta)}{h} = \frac{-\sin(\theta)}{1}\,, \tag{4}$$

where the ratios' denominators θ, h and 1 are all measured in radians. At this step, students can be invited to consider that the unit of measurement of the derivative of the cosine and sine functions is therefore the reciprocal radian.

It we calculate the first and second time-derivative of (2), we obtain the velocity and acceleration, which are commonly written as:

$$v(t) = \frac{dx}{dt}(t) = -\omega A\sin(\omega t+\varphi)\,, \tag{5}$$

$$a(t) = \frac{dv}{dt}(t) = -\omega^2 A\cos(\omega t+\varphi)\,. \tag{6}$$

If we ignore the previous considerations on the unit of measurement of the derivative of the cosine and sine functions, we get from (5) and (6) that the unit of measurement of the velocity v is the radian meter per second, and that of the acceleration a is the radian squared meter per second squared. These derivatives should be better considered as (with $\theta = \omega t + \varphi$):

$$v(t) = \frac{dx}{dt}(t) = \frac{dx}{d\theta}\frac{d\theta}{dx}(t) = -\omega A\frac{\sin(\omega t+\varphi)}{1}\,, \tag{7}$$

with 1 measured in radians and

$$a(t) = \frac{dv}{dt}(t) = \frac{dv}{d\theta}\frac{d\theta}{dt}(t) = -\omega^2 A\frac{\cos(\omega t+\varphi)}{1}\,, \tag{8}$$

with 1 measured in radians squared. In this way, the unit of measurement of the velocity v is the meter per second, and that of the acceleration a is the meter per second squared.

If we insert (2) and (8) in (1), we get:

$$-\omega^2 A\frac{\cos(\omega t+\varphi)}{1} = -\frac{k}{m}A\cos(\omega t+\varphi)\,, \tag{9}$$

with 1 measured in radians squared. After the simplification of the common factors, we get:

$$\frac{\omega^2}{1}=\frac{k}{m}, \qquad (10)$$

so that the angular frequency squared is now:

$$\omega^2 = 1\times\frac{k}{m}. \qquad (11)$$

The unit of measurement of the right side is now the radian squared per second squared, therefore the unit of measurement of the angular frequency is the radian per second and not the reciprocal second as derived in §II.

IV. THE FORCED OSCILLATOR WITH DAMPING

The more general case of a forced oscillator with damping is described by the following equation:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F, \qquad (12)$$

which is usually re-written as:

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2 x=\frac{F}{m}, \qquad (13)$$

where the damping constant γ is measured in the reciprocal second, and the proper angular frequency is actually the ratio $\omega_0/1$ measured in the reciprocal second.

If the driving force is an oscillatory function with angular frequency ω, (13) is more easily solved with the help of the complex exponential function as in Euler's formula:

$$e^{i\theta}=\cos(\theta)+i\sin(\theta). \qquad (14)$$

The exponential function exp(x) can be calculated only for real values of x which must be a physical quantity of dimension one and with no unit of measurement. Furthermore, exp(x) is also of dimension one and with no unit of measurement.

Since the result of a measure can only be expressed by a real (more precisely a rational) number, only real functions of the complex solution can be directly tested by an experiment. The right side of (14) shows that the unit of measurement of θ is the radian and that the imaginary unit i has no unit of measurement.

The analysis of the first and second derivatives of the complex exponential function introduces the same multiplicative factors seen in the case of the simple oscillator:

$$\frac{de^{i\theta}}{d\theta}=\lim_{h\to 0}\frac{e^{i(\theta+h)}-e^{i\theta}}{h}=\frac{ie^{i\theta}}{1}, \qquad (15)$$

with θ, h and 1 measured in radians, and

$$\frac{d^2e^{i\theta}}{d\theta^2}=\frac{i^2e^{i\theta}}{1}, \qquad (16)$$

with 1 measured in radians squared.

The complex solution of (13) is:

$$\hat{x}(t)=\frac{\hat{F}(t)}{m\left(\omega_0^2-\omega^2+i\gamma\omega\right)}. \qquad (17)$$

The real part of (17) is:

$$x(t)=\frac{F\cos(\omega t+\varphi)}{m\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}. \qquad (18)$$

As we have considered in (11), ω_0^2 is better written as $\omega_0^2/1$, ω^2 from the second derivative (16) as $\omega^2/1$, and $\gamma\omega$ from the first derivate (15) as $\gamma\omega/1$; they are all measured in the reciprocal second squared.

With these considerations on the derivatives, the students can combine together the units of measurement of all the physical quantities in (12) or (13).

V. ERROR PROPAGATION FOR OSCILLATING FUNCTIONS

Similar considerations can be done as for the calculations of error propagation. The most common textbooks on the statistical analysis of physical measurements (see, *e.g.*, [6, 7, 8]) are not usually concerned with units of measurement and dimensional analysis, besides stating that any physical quantity and its error must have the same physical dimension and unit of measurement.

The general formula for error propagation in the case of independent variables is:

$$\sigma_y=\sqrt{\sum_{n=1}^{N}\left(\frac{\partial y}{\partial x_n}(\bar{x}_1,\bar{x}_2,...,\bar{x}_N)\sigma_n\right)^2}, \qquad (19)$$

where x_1, x_2, …, x_N are the measured independent variables; σ_1, σ_2, …, σ_N are their respective errors; y is the dependent variable to be evaluated in the mean values of the independent variables, and σ_y is its calculated error. The case of dependent variables requires the use of a more general formula with the covariance of the independent variable. This fact is irrelevant in the problem under consideration.

Let us suppose that the functional relation contains a simple oscillatory function, such as $y(\theta) = \cos(\theta)$. If we measure the argument θ, we get a value of the argument with its error, both measured in radians. Eq. (19) gives the following error for the calculated variable y:

$$\sigma_y = \left|\sin(\bar{\theta})\right|\sigma_\theta . \quad (20)$$

With Eq. (20), the students must face the same problem seen in the previous paragraphs: the left side is the error of a cosine and has no unit of measurement; the right side is instead measured in radians. The calculation of the derivative of *y*, as done above, leads to the relation:

$$\sigma_y = \frac{\left|\sin(\bar{\theta})\right|}{1}\sigma_\theta , \quad (21)$$

with the function $-\sin(\theta)/1$ measured in the reciprocal radian. Both sides of (21) have now no unit of measurement.

VI. CONCLUSIONS

The analysis of the units of measurements in a step-by-step derivation of the oscillatory motion equations can help the students in understanding the role played by the radian. Even if it is a unit of measurement with dimension one, which does not cause any particular problem in the dimensional analysis of the physical quantities involved, an explicit mention of the radian should be considered when taking into account the units of measurement of derived quantities. An analysis of this kind can also help the students in considering from a physical point of view the derivative of the above-considered trigonometric functions as the limit of the incremental ratio of physical quantities, as it is usually done when treating velocity and acceleration in cinematic.

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