Thermodynamics of Planck oscillator in microcanonical ensemble

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Abstract

A Planck Oscillator [PO] can be considered to be a collection of harmonic oscillators with energy $\epsilon = \hbar\omega$ whit ω varying continuosly. When all the oscillators have the same frequency they are called Einstein Oscillators [EO]. In this short article we work out the thermodynamics [TD] of PO in Micro Canonical Ensemble [MCE].

Keywords: Planck Oscillator, Micro Canonical Ensemble.

Resumen

Un oscilador Planck [PO] puede ser considerado para ser una colección harmónica de osciladores con energía $\epsilon = \hbar\omega$ con ω variando continuamente. Cuando todos los osciladores tienen la misma frecuencia pueden ser llamados Osciladores de Einstein [EO]. En este pequeño artículo trabajamos con la termodinámica [TD] de PO en el Ensamble Micro Canonical [MCE].

Palabras clave: Oscilador Planck, Ensamble Micro Canonical.

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I. INTRODUCTION

Historically the black body distribution curve was explained by Planck using the concept of PO [1, 2]. Planck used Boltzmann technique for finding the entropy and from entropy he obtained energy of a single harmonic oscillator and then obtained the distribution curve using MCE. In textbooks the distribution law is obtained using canonical ensemble [3] or grand canonical ensemble [4]. The complete thermodynamics like total entropy, total energy, total pressure etc, using MCE is not available in any of the textbooks. When we teach statistical mechanics in Post Graduate classes students always raise this issue of nonavailability of the thermodynamics using MCE. This article attempts to clarify this issue.

II. ROUTE TO THERMODYNAMICS

The crucial problem in micro canonical ensemble formalism is the computation of the number of possible micro states Ω. For Planck Oscillator, there is no direct method to find Ω in MCE. For this first we had to find Ω for Einstein Oscillators and from this Ω entropy of EO must be found out using Boltzmann relation. From this entropy, S_{EO}, which is the entropy corresponding to system of N oscillators having same frequency, single Harmonic Oscillator (HO) entropy is computed using the extensive property of entropy. Then using First law of thermodynamics single HO energy (u) is calculated. Using this single HO energy all the thermodynamics of collection of oscillators with variable frequency are calculated which will give the TD of Planck Oscillator. It is observed that the knowledge about the relation P=1/3(U/V) is the pre requisite for calculating the pressure of the system. The TD obtained from CE and GCE are

$$\mathbf{S} = \frac{32\pi^5 k^4 T^3 V}{45 h^3 c^3}, \qquad (1)$$

$$U = \frac{8}{15}\frac{\pi^5 k^4 T^4 V}{h^3 c^3}, \qquad (2)$$

$$P = \frac{8\pi^5 k^4 T^4}{45 h^3 c^3}, \qquad (3)$$

where S, U and P are the entropy, energy and pressure. T is absolute temperature, V is the volume, h is Planck's constant and c is the velocity of light. We have to deduce the above expressions using MCE. The other TD quantities like Helmholtz free energy, specific heat etc can be obtained from the above TD quantities using TD relations.

III. SINGLE HO ENTROPY

Consider N HOs. The HO in the n_1 th state will emit n_1 quanta of energy. Hence the total energy

$$E = n_1\hbar\omega + n_2\hbar\omega + \ldots + n_N\hbar\omega = M\hbar\omega, \tag{4}$$

where M is the total number of states which will be the quanta of energy emitted by N HOs. Thus the number of ways in which M particles can be emitted by N harmonic oscillators is given by

$$\Omega = \frac{(M+N-1)!}{M!(N-1)!}. \tag{5}$$

Since N»1

$$\Omega = \frac{(\frac{E}{\hbar\omega}+N)!}{(\frac{E}{\hbar\omega})!N!}, \tag{6}$$

$$\ln\Omega = \ln\left(\frac{E}{\hbar\omega}+N\right)! - \ln\left(\frac{E}{\hbar\omega}\right)! - \ln N!. \tag{7}$$

For large N we can use stirling's approximation

$$\ln N! = N\ln N - N. \tag{8}$$

Thus

$$\ln\Omega = \left(\frac{E}{\hbar\omega}+N\right)\ln\left(\frac{E}{\hbar\omega}+N\right) - \left(\frac{E}{\hbar\omega}\right)\ln\left(\frac{E}{\hbar\omega}\right) - N\ln N. \tag{9}$$

Our interest is to get the energy of a single HO. Let u be single HO energy. All the HO have the same energy. Then the total energy E of N HOs will be

$$E = uN. \tag{10}$$

Substituting this in the previous expression we get

$$\ln\Omega = \left(\frac{uN}{\hbar\omega}+N\right)\ln\left(\frac{uN}{\hbar\omega}+N\right) - \left(\frac{uN}{\hbar\omega}\right)\ln\left(\frac{uN}{\hbar\omega}\right)$$

$$-N\ln N. \tag{11}$$

Taking N outside $\ln\Omega$ is

$$= N\left(\frac{u}{\hbar\omega}+1\right)\ln N\left(\frac{u}{\hbar\omega}+1\right) - N\left(\frac{u}{\hbar\omega}\right)\ln N\left(\frac{u}{\hbar\omega}\right)$$

$$-N\ln N. \tag{12}$$

Using Boltzmann relation for entropy [4]

$$\mathbf{S}_{EO} = k\ln\Omega. \tag{13}$$

Where $\mathbf{S}_{EO}$ is the entropy of oscillator with constant ω.

$$\mathbf{S}_{EO} = Nk\left[\left(1+\frac{u}{\hbar\omega}\right)\ln\left(1+\frac{u}{\hbar\omega}\right) - \left(\frac{u}{\hbar\omega}\right)\ln\left(\frac{u}{\hbar\omega}\right)\right]. \tag{14}$$

For a single HO the entropy is

$$S_{EO} = k\left[\left(1+\frac{u}{\hbar\omega}\right)\ln\left(1+\frac{u}{\hbar\omega}\right) - \left(\frac{u}{\hbar\omega}\right)\ln\left(\frac{u}{\hbar\omega}\right)\right]. \tag{15}$$

The difference between two entropies must be taken into consideration (one is represented in bold letter).

IV. ENERGY OF A SINGLE HO

From I law of thermodynamics

$$Td\mathbf{S}_{EO} = dE + PdV, \tag{16}$$

$$\left(\frac{\partial\mathbf{S}_{EO}}{\partial E}\right)_V = \frac{1}{T}. \tag{17}$$

For a single HO

$$\left(\frac{\partial S_{EO}}{\partial u}\right)_V = \frac{1}{T}, \tag{18}$$

$$\frac{1}{kT} = \frac{1}{\hbar\omega}\left[\ln\left(1+\frac{u}{\hbar\omega}\right) - \ln\left(\frac{u}{\hbar\omega}\right)\right]. \tag{19}$$

Rearranging

$$u = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}}-1}. \tag{20}$$

This is the energy of single HO.

V. TOTAL ENERGY FOR FREQUENCY FROM 0 TO ∞

Now we have to find the total energy of PO. For PO frequency varies from 0 to ∞. From the basic definition in SM the number of state between ω and $\omega + d\omega$ [4]

$$g(\omega)\, d\omega = \frac{V\omega^2\, d\omega}{2\pi^2 c^3}. \tag{21}$$

Due to two types of polarization of the quanta emitted, the internal degree of freedom $g_I = 2$.

$$U = g_I \int_0^\infty g(\omega) d\omega\, u(\omega), \tag{22}$$

$$U = 2\int_0^\infty g(\omega) d\omega \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}, \tag{23}$$

$$= 2\int_0^\infty \frac{V\omega^2 d\omega}{2\pi^2 c^3} \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1},$$

Putting

$$\frac{\hbar\omega}{kT} = x,$$

and using the standard integral $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$, (24)

$$U = \frac{8}{15} \frac{\pi^5 k^4 T^4 V}{h^3 c^3}. \tag{25}$$

A. Total Entropy for frequency from 0 to ∞

It is difficult to integrate the previous equation for entropy to get total entropy. So it is modified as below Substituting for u in Eq. (15)

$$S_{EO} = k\left[\frac{e^{\frac{\hbar\omega}{kT}}}{e^{\frac{\hbar\omega}{kT}} - 1} \ln\left(\frac{e^{\frac{\hbar\omega}{kT}}}{e^{\frac{\hbar\omega}{kT}} - 1}\right) - \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1} \ln\left(\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}\right)\right]. \tag{26}$$

Simplifying

$$S_{EO} = k\left[-\ln\left(1 - e^{\frac{-\hbar\omega}{kT}}\right) + \frac{\frac{\hbar\omega}{kT}}{e^{\frac{\hbar\omega}{kT}} - 1}\right]. \tag{27}$$

Using $\ln(1-x) = \sum_{l=1}^{\infty} (-1) \frac{x^l}{l}$,

$$S_{EO} = k\left[\frac{\frac{\hbar\omega}{kT}}{e^{\frac{\hbar\omega}{kT}} - 1} - \sum_{l=1}^{\infty} (-1) \frac{e^{\frac{-\hbar\omega}{kT} l}}{l}\right], \tag{28}$$

$$S_{EO} = k\left[\frac{\frac{\hbar\omega}{kT}}{e^{\frac{\hbar\omega}{kT}} - 1} + \sum_{l=1}^{\infty} \frac{1}{l} e^{\frac{-\hbar\omega}{kT} l}\right]. \tag{29}$$

Total entropy

$$\mathbf{S} = g_I \int_0^\infty S_{EO}(\omega)\, g(\omega) d\omega, \tag{30}$$

$$= 2\int_0^\infty \frac{V\omega^2 d\omega}{2\pi^2 c^3} k\left[\frac{\frac{\hbar\omega}{kT}}{e^{\frac{\hbar\omega}{kT}} - 1} + \sum_{l=1}^{\infty} \frac{1}{l} e^{\frac{-\hbar\omega}{kT} l}\right]. \tag{31}$$

Simplifying

$$= \frac{8\pi k^4 T^3 V}{h^3 c^3}\left[\frac{\pi^4}{15} + \sum_{l=1}^{\infty} \frac{1}{l} 2!\, l^{-3}\right]. \tag{32}$$

Using $\sum_{l=1}^{\infty} \frac{1}{l^4} = \frac{\pi^4}{90}$, (33)

$$= \frac{8\pi k^4 T^3 V}{h^3 c^3}\left[\frac{\pi^4}{15} + 2.\frac{\pi^4}{90}\right],$$

$$\mathbf{S} = \frac{32\pi^5 k^4 T^3 V}{45 h^3 c^3}. \tag{34}$$

B. Total Pressure

From I law of TD

$$Td\mathbf{S} = dU + PdV. \tag{35}$$

From Electrodynamics and TD we know that for radiation

$$P = \frac{1}{3}\frac{U}{V}. \tag{36}$$

Hence

$$Td\mathbf{S} = d(3PV) + P\, dV = 4PdV + 3V\, dP. \tag{37}$$

Thus

$$\left(\frac{\partial \mathbf{S}}{\partial V}\right)_P = 4\frac{P}{T}. \tag{38}$$

Hence pressure is

$$P = \frac{8\pi^5 k^4 T^4}{45h^3c^3}. \tag{39}$$

II. CONCLUSION

Thus in MCE the TD of PO is obtained. The pressure is obtained using the already available relation $P = \frac{1}{3}\frac{U}{V}$. But in grand canonical ensemble and canonical ensemble this previous knowledge and also the indirect derivation is not necessary. The pressure energy relation is automatically established [3, 4]. In MCE this relation has to be used and only then we will get the pressure. This may be the reason for not deriving the TD of PO in textbooks using MCE.

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