The Landau-Lifshitz equation for a central field in General Relativity

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(Received 22 March 2012, accepted 28 June 2012)

Abstract
The equation of motion for a spinless point charged particle is analyzed for the central field case in General Relativity, by considering reaction force due to the radiation of the charge. It is started by generalizing the Landau-Lifshitz equation in Special Relativity to General Relativity by using the equivalence principle. The result is different of the equation obtained by Hobbs by two terms; one of them related with the curvature of space-time and the other a tail term. These two terms are going to be neglected. In order to consider the effects of mass and charge in space the Reissner-Nordström metric is used. A set of three equations are found to solve the problem which are similar to the ones obtained in Classical Electrodynamics.

Keywords: Central force, Landau-Lifshitz equation, Reissner-Nordström metric.

Resumen
Considerando la fuerza de reacción a la radiación, se analiza la ecuación de movimiento para una partícula puntual cargada y sin espín en el caso del campo central en relatividad general. Se inicia generalizando la ecuación de Landau-Lifshitz en relatividad especial a relatividad general por medio del principio de equivalencia. El resultado difiere de la ecuación obtenida por Hobbs por dos términos; uno relacionado con la curvatura del espacio tiempo y el otro un término de cola. Estos términos se despreciarán. Se utiliza la métrica de Reissner-Nordström para considerar los efectos de la masa y la carga en el espacio. Se encuentra un conjunto de tres ecuaciones para poder resolver el problema, que es similar al obtenido en el caso de la electrodinámica clásica.

Palabras clave: Fuerza central, ecuación de Landau-Lifshitz, métrica de Reissner-Nordström.

PACS: 04, -04.25.g, 04.25.Nx. **ISSN 1870-9095**

I. INTRODUCTION

An old problem in physics is the radiation reaction phenomena and the self-force due to accelerated charges [1]. Although this is an unsolved matter, nowadays it has a great importance because it has been reached high energies beams that require the search of an equation that is physically correct [1, 2]. The most known equations that include radiation terms are the Lorentz-Dirac equation, the Landau-Lifshitz (LL) equation and the Ford O´Connell equation, among others [1, 3, 4]. So many authors have studied the validity of these known equations or even proposed new ones. So it is clear that the problem remains to be finding a good equation that includes the motion of the charged particle and its radiation. Even if many authors consider that the LL equation possesses many objections, it is considered by many others as the correct equation of motion of a spinless point charged particle [5, 6, 7]. Following these ideas, the LL equation is chosen to develop it in General Relativity.

However if it is supposed that the correct equation is known, a new problem arises. The known equation is written for a Minkowski space-time, but it is necessary to know it in General Relativity, in order to apply it to some astrophysical phenomena. The immediate idea consists of using the equivalence principle of General Relativity which states the form to change an equation in a Minkowski or almost plane space to a general curved space-time. However many authors claim that this cannot be done because this principle is not valid for charges, unless we talk about apparent gravitational fields [8]. This problem is faced in brilliant ways by DeWitt and Brehme [9] corrected by Hobbs [10], and also by Quinn and Wald [11] via an axiomatic approach, and also by Villaroel [12] and López-Bonilla [13], among others.

If the equivalence principle is accepted and in consequence the covariance principle is applied in a straight forward manner into the LL equation, a generalized equation is deduced. However, if we compare it with Hobbs Eq. [10] just two terms are missed. These two terms are related with the curvature of the space and with a tail term that comes from the past of the particle. In this article, we will not discuss the validity of both equations and the two terms are going to be ignored. So, once the LL in a general curved space time is obtained, it is going to be applied to the case of a central field composed by a charged massive particle centered at the origin and a test charged particle that is going to be radiating.

The paper is organized as follows: In Sec. II, the LL equation is described. Next, in Sec. III, the equivalence principle is discussed. Sec. IV is advocated to obtain the LL equation generalization to curved space time. Finally, in Sec. V, the equations of motion for a radiating particle in central field are obtained. In Sec. VI some concluding remarks are given.

II. LANDAU-LIFSCHITZ EQUATION

The LL equation is one of the most accepted equations describing the motion of a charge that radiates due to its acceleration [1]. Indeed, it is a second order differential equation which possesses neither runaway solutions nor preaccelerations, the self-force is null when the external force and its first derivative are zero, and it satisfies the inertia principle [3]. Also, it has been pointed out that it admits exact solutions for appropriate external fields, however it implies that the Larmor formula must be modified [4].

The LL equation in special relativity is given by

$$ma^{\mu} = \frac{q}{c^2}F^{\mu\nu}u_{\nu} + \tau_0\left[\frac{q}{c^2}\left\{\frac{\partial F^{\mu\nu}}{\partial x^{\alpha}}v^{\alpha}v_{\nu} - \frac{q}{c^2}F^{\mu\nu}F_{\alpha\nu}v^{\alpha}\right\} + \right.$$
$$q2mc4F2v\mu, \qquad (1)$$

where q and m represent the charge and the mass of the test particle respectively. $F^{\mu\nu}$ is the electromagnetic field tensor given by

$$F^{\mu\nu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} - \frac{\partial A^{\nu}}{\partial x^{\mu}}\ , \qquad (2)$$

with A^{μ} the 4- electromagnetic potential

$$A^{\mu} = (\phi, \boldsymbol{A}). \qquad (3)$$

Also, τ_0 is known as the characteristic time and is given by

$$\tau_0 = \frac{kq^2}{mc^3}, \qquad (4)$$

being q, m and c the charge, the mass and the speed of light respectively. It has to be pointed out that some authors claim the validity of the LL equation when a charged particle with structure is considered [14]. LL equation will be generalized to General Relativity and analyzed for the central field case.

III. EQUIVALENCE PRINCIPLE

The principle of equivalence has many versions, but it principally states that gravitational forces and inertial forces acting on a body are equivalent and are indistinguishable from each other. [15]. It says that at every space-time point in an arbitrary gravitational field it is possible to choose a locally coordinate system, such that, within a sufficiently small neighborhood of the point in question, the laws of nature take the same form as in unaccelerated coordinate systems in the absence of gravitation. In such locally coordinate system, the laws of nature have the form given by special relativity. By a sufficiently small neighborhood of the point it is meant a small region which contains the point itself and where the gravitational field could be taken constant through it [16].

Generally, two different versions are accepted: the weak principle and the strong one. The weak principle is obtained replacing the laws of nature by laws of motion of freely falling particles. The strong principle is the principle upon which general relativity theory is based and it is applied to all laws of nature [8, 15, 16].

There is an alternative equivalence principle that is known as the principle of general covariance. It states that a physical equation is valid in a general gravitational field if the equation satisfies two conditions. The first condition requires that the equation holds in absence of gravitation, which means that it agrees with the laws of special relativity when the metric tensor is replaced by the Minkowski tensor and in consequence the Christoffel's symbols vanish. The second condition is that the equation must be generally covariant, in other words, the equation must maintain its form when a transformation of coordinates is applied [16].

However, Rohrlich [8] explains that if the equivalence principle is going to be used in the case of charged particles, the difference between the weak and strong formulation seems to be important. He claims that, since the weak principle is a statement about the simulation of certain gravitational fields by acceleration and is valid in Newtonian physics as well as in special relativity, it must be valid for charged particles. However the strong formulation makes a non-trivial statement about particles in general gravitational fields, so it appears to be not valid for radiating particles.

Nevertheless, if the covariance principle criteria are considered into the LL equation, it is possible to apply the equivalence principle and in consequence transform it to an equation of motion for charged radiating particles in general relativity.

IV. THE LANDAU-LIFSHITZ EQUATION IN GENERAL RELATIVITY

By using the equivalence and covariance principle it is possible to write the L-L equation in a general curved space, by simply substituting the common derivatives by covariant derivatives. First it has to be noticed that the electromagnetic 4-potential A^μ does not change. Also the electromagnetictensor is transformed as

$$F^{\mu\nu} = A^{\mu}{}_{;\nu} - A^{\nu}{}_{;\mu} = \frac{\partial A^\mu}{\partial x^\nu} - \frac{\partial A^\nu}{\partial x^\mu}, \qquad (5)$$

due to the skew-symmetry of the electromagnetic tensor. Other thing that has to be transformed, is the 4-acceleration as

$$a^\mu = \frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta, \qquad (6)$$

with u^α the 4-velocity. Also the derivatives of the electromagnetic tensor are transformed as

$$F^{\mu\nu}{}_{;\alpha} = \frac{\partial F^{\mu\nu}}{\partial x^\alpha} + \Gamma^\mu_{\alpha\rho} F^{\rho\nu} + \Gamma^\nu_{\alpha\rho} F^{\mu\rho}, \qquad (7)$$

where $\Gamma^{\mu}{}_{\alpha\rho}$ are the Christoffel symbols. So the LL equation in General Relativity is

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\beta\gamma} u^\beta u^\gamma = \frac{q}{mc^2} F^{\mu\nu} u_\nu + \tau_0 \left[\frac{q}{mc^2}\left\{\left(\frac{\partial F^{\mu\nu}}{\partial x^\alpha} + \Gamma^\mu_{\alpha\rho} F^{\rho\nu} + \Gamma \alpha\rho\nu F\mu\rho v\alpha v\nu - qmc22 F\mu\nu F\alpha\nu v\alpha - F2v\mu. \right.\right.\right. \qquad (8)$$

Notice that this technique avoids the curvature and tail terms of Hobbs Eq. [10].

V. THE CENTRAL FORCE FIELD WITH RADIATION

The curved space time by a charged and massive particle with spherical symmetry fixed in the origin is represented by the Reissner-Nordström metric, which is

$$ds^2 = \xi dct^2 - \frac{1}{\xi} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2, \qquad (9)$$

with

$$\xi = 1 - \frac{2GM}{c^2 r} + \frac{GkQ^2}{c^4 r^2}, \qquad (10)$$

where G and k are the gravitational constant and the coulomb constant respectively, M and Q are the mass and charge of the body. For this metric, the Christoffel's symbols that are different from zero are in Table I.

TABLE I. Christoffel's symbols for the Reissner-Nordstrom metric.

$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2\xi}\frac{d\xi}{dr}$	$\Gamma^1_{00} = \frac{\xi}{2}\frac{d\xi}{dr}$	$\Gamma^1_{11} = -\frac{1}{2\xi}\frac{d\xi}{dr}$
$\Gamma^1_{22} = -r\xi$	$\Gamma^1_{33} = -r\xi\sin^2\theta$	$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$
$\Gamma^2_{33} = -\sin\theta\cos\theta$	$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$	$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$

In the case of a central field due to a static charge Q, the 4-electromagnetic potential is

$$A_\mu = \left(\frac{kQ}{r}, \mathbf{0}\right). \qquad (11)$$

Therefore the components of the electromagnetic tensor that does not vanishes are

$$F_{01} = -F_{10} = -\frac{kQ}{r^2}. \qquad (12)$$

Now it is possible to get the equations of motion for a central field due to charge, considering radiation from a test charge, obtaining a set of four equations, which are

$$\frac{d^2ct}{ds^2} + \frac{1}{\xi}\frac{d\xi}{dr}\frac{dr}{ds}\frac{dct}{ds} = -\frac{kQq}{mc^2r^2}\frac{1}{\xi}\frac{dr}{ds} + \tau_0\left[\frac{2kQq}{mc^2r^3}\frac{1}{\xi}\left(\frac{dr}{ds}\right)^2 - rd\theta ds2 - r\sin2\theta d\varphi ds2 - kQqmc2r22dctds, \right. \qquad (13)$$

$$\frac{d^2r}{ds^2} + \frac{\xi}{2}\frac{d\xi}{dr}\left(\frac{dct}{ds}\right)^2 - \frac{1}{2\xi}\frac{d\xi}{dr}\left(\frac{dr}{ds}\right)^2 - r\xi\left(\frac{d\theta}{ds}\right)^2 - r\xi\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2 = -\frac{kQq}{mc^2r^2}\xi\frac{dct}{ds} + \tau_0\left[\frac{2kQ}{mc^2r^3}\xi\frac{dct}{ds}\frac{dr}{ds} - kQqmc2r22drds, \right. \qquad (14)$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 = -\tau_0\left[\frac{kQq}{mc^2r^3}\xi\frac{dct}{ds}\frac{d\theta}{ds} + 2kQqmc2r22d\theta ds, \right. \qquad (15)$$

$$\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} + \cot\theta\frac{d\theta}{ds}\frac{d\varphi}{ds} = -\tau_0\left[\frac{kQq}{mc^2r^3}\xi\frac{dct}{ds}\frac{d\varphi}{ds} + 2kQqmc2r22d\varphi ds. \right. \qquad (16)$$

In the classical Newtonian and relativistic central field, it is considered that the motion of the particle is in a plane, specifically in $\theta = \pi/2$, which is also valid in this case. Therefore, the set of Eqs. (13), (14), (15) and (16) is reduced to the following set of three equations

$$\frac{d^2ct}{ds^2} + \frac{1}{\xi}\frac{d\xi}{dr}\frac{dr}{ds}\frac{dct}{ds} = -\frac{kQq}{mc^2r^2}\frac{1}{\xi}\frac{dr}{ds} + \tau_0,$$

$$\left[\frac{2kQq}{mc^2r^3}\frac{1}{\xi}\left(\frac{dr}{ds}\right)^2 - r\left(\frac{d\varphi}{ds}\right)^2 - \left(\frac{kQq}{mc^2r^2}\right)^2\frac{dct}{ds}\right], \qquad (17)$$

$$\frac{d^2r}{ds^2} + \frac{\xi}{2}\frac{d\xi}{dr}\left(\frac{dct}{ds}\right)^2 - \frac{1}{2\xi}\frac{d\xi}{dr}\left(\frac{dr}{ds}\right)^2 - r\xi\left(\frac{d\varphi}{ds}\right)^2 = -\frac{kQq}{mc^2r^2}\xi\frac{dct}{ds} + \tau_0\left[\frac{2kQ}{mc^2r^3}\xi\frac{dct}{ds}\frac{dr}{ds} - \left(\frac{kQq}{mc^2r^2}\right)^2\frac{dr}{ds}\right], \qquad (18)$$

$$\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} = -\tau_0\left[\frac{kQq}{mc^2r^3}\xi\frac{dct}{ds}\frac{d\varphi}{ds} + 2\left(\frac{kQq}{mc^2r^2}\right)^2\frac{d\varphi}{ds}\right] \qquad (19)$$

Also, it should be remembered that the angular momentum per unit mass for a central field is given by

$$\frac{d\varphi}{ds}=\frac{h}{r^2}. \tag{20}$$

So the last set of equations can be rewritten as

$$\frac{d^2ct}{ds^2}+\frac{1}{\xi}\frac{d\xi}{dr}\frac{dr}{ds}\frac{dct}{ds}=-\frac{kQq}{mc^2r^2}\frac{1}{\xi}\frac{dr}{ds}+\tau_0\left[\frac{2kQq}{mc^2r^3}\frac{1}{\xi}\left(\frac{dr}{ds}\right)^2-\right. \\ h2r3-kQqmc2r22dctds, \tag{21}$$

$$\frac{d^2r}{ds^2}+\frac{\xi}{2}\frac{d\xi}{dr}\left(\frac{dct}{ds}\right)^2-\frac{1}{2\xi}\frac{d\xi}{dr}\left(\frac{dr}{ds}\right)^2-\xi\frac{h^2}{r^3}=-\frac{kQq}{mc^2r^2}\xi\frac{dct}{ds}+ \\ \tau_0\left[\frac{2kQ}{mc^2r^3}\xi\frac{dct}{ds}\frac{dr}{ds}-\left(\frac{kQq}{mc^2r^2}\right)^2\frac{dr}{ds}\right], \tag{22}$$

$$\frac{d}{ds}\left(\frac{h}{r^2}\right)+\frac{2}{r}\frac{dr}{ds}\frac{h}{r^2}=-\tau_0\left[\frac{kQq}{mc^2r^3}\frac{h}{r^2}\xi\frac{dct}{ds}+2\left(\frac{kQq}{mc^2r^2}\right)^2\frac{h}{r^2}\right]. \tag{23}$$

This set of equations is reduced to the set of equations of the central field with charges in general relativity if τ_0 =0, that is, by neglecting the radiation of the electric charge. Following Hammond comment [17] about Astrophysical Applications, ("… with is compact astrophysical sources such as neutron stars or magnetars, and black holes. In addition to the electromagnetic force there is also the gravitational force. The equation of motion for a charged particle in a combined gravitational and electromagnetic field with radiation reaction is"), when a charged particle is submitted to a central field electric field, the equations with describes the motion of a charge particle are Eqs. (17), (18) and (19).

VI. CONCLUSIONS

In this paper the equations of motion for a radiating charge in central fields, both electric and gravitational, are obtained by means of the LL equation. It has been considered that the equivalence principle can be applied in a straight forward manner, avoiding the discussion about the curvature and tail Hobbs terms.

The equations obtained are similar to those obtained for a central field but in Special Relativity [14], the difference is that some terms of the equations of motion in General Relativity has a factor ξ that depends of the chosen metric. Also if the radiation is neglected the equations of motion for a central field with the Reissner-Nördstrom metric are recovered. The solutions of those equations are rosettes, which suggest that perhaps in the case of radiation a precession may appear.

It can be observed that the obtained equations are very complex, so, it is necessary to apply numerical methods in order to get a solution.

ACKNOWLEDGEMENTS

This work was partially supported by COFFA, EDI, IPN and CONACYT.

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