Inextensible flows of $\mathcal{S}-\mathrm{s}$ surfaces of biharmonic $\mathcal{S}$ -curves according to Sabban frame in Heisenberg Group Heis^3

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Abstract

In this paper, we study inextensible flows of $\mathcal{S}-\mathbf{s}$ surfaces according to Sabban frame in the Heisenberg group Heis^3. We characterize the biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Heisenberg group Heis^3. Finally, we find explicit parametric equations of one parameter family of $\mathcal{S}-\mathbf{s}$ surfaces according to Sabban Frame.

Keywords: Energy, Bienergy, Biharmonic curve, Heisenberg group, **s** surface.

Resumen

En este trabajo, estudiamos los flujos inextensibles de $\mathcal{S}-\mathbf{s}$ superficies de acuerdo al marco de Sabban en el grupo Heis^3 de Heisenberg. Caracterizamos las curvas biarmónicas en términos de su curvatura geodésica y demostramos que todas las curvas biarmónicas son hélices en el Heis^3 grupo de Heisenberg. Finalmente, encontramos ecuaciones paramétricas explícitas de familia de parámetro uno de superficies $\mathcal{S}-\mathbf{s}$ que concuerdan con el marco de Sabban.

Palabras clave: Energía, Bienergía, curva biarmónica, grupo de Heisenberg, superficie **s**.

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I. INTRODUCTION

Physically, inextensible curve and surface flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications.

Firstly, harmonic maps are given as follows:

Harmonic maps $f:(M,g)\longrightarrow(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1}$$

and they are therefore the solutions of the corresponding Euler-Lagrange equation [1, 2, 3, 4]. This equation is given by the vanishing of the tension field

$$\tau(f)=\operatorname{trace}\nabla df\ . \tag{2}$$

Secondly, biharmonic maps are given as follows:

The bienergy of a map *f* by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [5], showing that the Euler-Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f)&=-\mathcal{J}^f(\tau(f))=-\Delta\tau(f)-\operatorname{trace}R^N(df,\tau(f))df\\&=0,\end{aligned} \tag{4}$$

where $\mathcal{J}^f$ is the Jacobi operator of *f*. The equation $\tau_2(f)=0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic [6, 7].

In this paper, we study inextensible flows of $\mathcal{S}-\mathbf{s}$ surfaces according to Sabban frame in the Heisenberg group Heis^3. We characterize the biharmonic curves in terms of their geodesic curvature and we prove that all of \ biharmonic curves are helices in the Heisenberg group Heis^3. Finally, we find explicit parametric equations of one parameter family of $\mathcal{S}-\mathbf{s}$ surfaces according to Sabban Frame.

II. THE HEISENBERG GROUP HEIS^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}). \quad (5)$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group [8].

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial x},\ \mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\ \mathbf{e}_3=\frac{\partial}{\partial z}, \quad (6)$$

for which we have the Lie products

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,[\mathbf{e}_2,\mathbf{e}_3]=[\mathbf{e}_3,\mathbf{e}_1]=0 \ .$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1.$$

We obtain

$$\begin{aligned}
\nabla_{\mathbf{e}_1}\mathbf{e}_1&=\nabla_{\mathbf{e}_2}\mathbf{e}_2=\nabla_{\mathbf{e}_3}\mathbf{e}_3=0,\\
\nabla_{\mathbf{e}_1}\mathbf{e}_2&=-\nabla_{\mathbf{e}_2}\mathbf{e}_1=\frac{1}{2}\mathbf{e}_3,\\
\nabla_{\mathbf{e}_1}\mathbf{e}_3&=\nabla_{\mathbf{e}_3}\mathbf{e}_1=-\frac{1}{2}\mathbf{e}_2,\\
\nabla_{\mathbf{e}_2}\mathbf{e}_3&=\nabla_{\mathbf{e}_3}\mathbf{e}_2=\frac{1}{2}\mathbf{e}_1.
\end{aligned} \quad (7)$$

The components $\{R_{ijkl}\}$ of R relative to $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are defined by

$$R_{ijk}=R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\ R_{ijkl}=R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l)=g\big(R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_l,\mathbf{e}_k\big).$$

The non vanishing components of the above tensor fields are

$$R_{121}=\frac{3}{4}\mathbf{e}_2,\ R_{131}=-\frac{1}{4}\mathbf{e}_3,\ R_{122}=-\frac{3}{4}\mathbf{e}_1,$$

$$R_{232}=-\frac{1}{4}\mathbf{e}_3,\ R_{133}=\frac{1}{4}\mathbf{e}_1,\ R_{233}=\frac{1}{4}\mathbf{e}_2,$$

and

$$R_{1212}=-\frac{3}{4},\ R_{1313}=R_{2323}=\frac{1}{4}.$$

III. BIHARMONIC $\mathcal{S}$-CURVES ACCORDING TO SABBAN FRAME IN THE HEISENBERG GROUP HEIS^3

Let $\gamma: I\to Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis [9, 10, 11]. Then, we have the following Frenet formulas:

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T}&=\kappa\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{N}&=-\kappa\mathbf{T}+\tau\mathbf{B},\\
\nabla_{\mathbf{T}}\mathbf{B}&=-\tau\mathbf{N},
\end{aligned} \quad (8)$$

Where k is the curvature of γ and τ is its torsion [12],

$$g(\mathbf{T},\mathbf{T})=1,\ g(\mathbf{N},\mathbf{N})=1,\ g(\mathbf{B},\mathbf{B})=1,$$

$$g(\mathbf{T},\mathbf{N})=g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.$$

Now we give a new frame different from Frenet frame, [12, 13, 14, 15, 16]. Let $\alpha: I\to\mathbb{S}^2_{Heis^3}$ be unit speed spherical curve. We denote σ as the arc-length parameter of α. Let us denote $\mathbf{t}(\sigma)=\alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of α. We now set a vector $\mathbf{s}(\sigma)=\alpha(\sigma)\times\mathbf{t}(\sigma)$ along α. This frame is called the Sabban frame of α on the Heisenberg group Heis^3. Then we have the following spherical Frenet-Serret formulae of α:

$$\begin{aligned}
\nabla_{\mathbf{t}}\alpha&=\mathbf{t},\\
\nabla_{\mathbf{t}}\mathbf{t}&=-\alpha+\kappa_g\mathbf{s},\\
\nabla_{\mathbf{t}}\mathbf{s}&=-\kappa_g\mathbf{t},
\end{aligned} \quad (9)$$

where κ_g is the geodesic curvature of the curve α on the $\mathbb{S}^2_{Heis^3}$ and

$$g(\mathbf{t},\mathbf{t})=1,\ g(\alpha,\alpha)=1,\ g(\mathbf{s},\mathbf{s})=1,$$

$$g(\mathbf{t},\alpha)=g(\mathbf{t},\mathbf{s})=g(\alpha,\mathbf{s})=0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\alpha=\alpha_1 e_1+\alpha_2\mathbf{e}_2+\alpha_3\mathbf{e}_3,$$

$$\mathbf{t}=t_1\mathbf{e}_1+t_2\mathbf{e}_2+t_3\mathbf{e}_3, \tag{10}$$

$$\mathbf{s}=s_1\mathbf{e}_1+s_2\mathbf{e}_2+s_3\mathbf{e}_3.$$

To separate a biharmonic curve according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $\mathcal{S}$-curve.

Theorem 3.1. $\alpha: I™\, \mathbb{S}^2_{Heis^3}$ *is a biharmonic* $\mathcal{S}$-curve *if and only if*

$$\kappa_g=\text{constant}\neq 0,$$

$$1+\kappa_g^2=-[\frac{1}{4}-s_3^2]+\kappa_g[-\alpha_3 s_3], \tag{11}$$

$$\kappa_g''-\kappa_g^3=\alpha_3 s_3+\kappa_g[\frac{1}{4}-\alpha_3^2].$$

Proof. Using (5) and Sabban formulas (9), we have (11).

Lemma 3.2. ([9]) $\alpha: I™\, \mathbb{S}^2_{Heis^3}$ *is a biharmonic* $\mathcal{S}$-*curve if and only if*

$$\kappa_g=\text{constant}\neq 0,$$
$$1+\kappa_g^2=-[\frac{1}{4}-s_3^2]+\kappa_g[-\alpha_3 s_3], \tag{12}$$
$$\kappa_g^3=-\alpha_3 s_3-\kappa_g[\frac{1}{4}-\alpha_3^2].$$

Then the following result holds.

Theorem 3.3. ([9]), *All of biharmonic* $\mathcal{S}$-*curves in* $\mathbb{S}^2_{Heis^3}$ *are helices.*

IV. INEXTENSIBLE FLOWS OF $\mathcal{S}-\mathbf{s}$ SURFACES OF BIHARMONIC $\mathcal{S}$-CURVES ACCORDING TO SABBAN FRAME IN THE HEISENBERG GROUP HEIS^3

To separate a **s** surface according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathcal{S}-\mathbf{s}$ surface.

The purpose of this section is to study $\mathcal{S}-\mathbf{s}$ surfaces of biharmonic $\mathcal{S}$-curve in the Heisenberg group Heis^3.

The $\mathcal{S}-\mathbf{s}$ surface of γ is a ruled surface

$$\mathcal{P}^{\mathcal{S}}(\sigma,u)=\alpha(\sigma)+u\mathbf{s}(\sigma). \tag{13}$$

Definition 4.1. *A surface evolution* $\mathcal{P}^{\mathcal{S}}(\sigma,u,t)$ *and its flow* $\frac{\partial\mathcal{P}^{\mathcal{S}}}{\partial t}$ *are said to be inextensible if its first fundamental form* $\{\mathbf{E},\mathbf{F},\mathbf{G}\}$ *satisfies*

$$\frac{\partial\mathbf{E}}{\partial t}=\frac{\partial\mathbf{F}}{\partial t}=\frac{\partial\mathbf{G}}{\partial t}=0. \tag{14}$$

Definition 4.2. *We can define the following one-parameter family of developable ruled surface*

$$\mathcal{P}^{\mathcal{S}}(\sigma,u,t)=\alpha(\sigma,t)+u\mathbf{s}(\sigma,t). \tag{15}$$

Hence, we have the following theorem.

Theorem 4.3. *Let* $\mathcal{P}^{\mathcal{S}}$ *be one-parameter family of the* $\mathcal{S}-\mathbf{s}$ *surface of a unit speed non-geodesic biharmonic* $\mathcal{S}$-*curve. Then* $\frac{\partial\mathcal{P}^{\mathcal{S}}}{\partial t}$ *is inextensible if and only*

$$\frac{\partial}{\partial t}[(1-\kappa_g(t))\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]]^2$$

$$+\frac{\partial}{\partial t}[(1-\kappa_g(t))\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]]^2+\frac{\partial}{\partial t}[(1-\kappa_g(t))\cos\mathcal{E}(t)]^2=0,$$

$$\frac{\partial}{\partial t}[\frac{1}{\kappa_g(t)}[\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]$$

$$(\mathcal{M}(t)+\cos\mathcal{E}(t))-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_2(t)]]^2$$

$$+\frac{\partial}{\partial t}[\frac{1}{\kappa_g(t)}[-\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)](\mathcal{M}(t)+\cos\mathcal{E}(t))$$

$$+\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_3]]^2+\frac{\partial}{\partial t}[\frac{1}{\kappa_g(t)}[\cos\mathcal{E}(t)\sigma \tag{16}$$

$$-\frac{\mathcal{V}(t)\sigma+\mathcal{M}_1(t)}{2\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)-\frac{\sin 2[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]}{4\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)$$

$$-[\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_3(t)][-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]$$

$$+\mathcal{M}_2(t)]+\frac{\mathcal{M}_2(t)}{\mathcal{V}(t)}\sin^3\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_4]]^2=0,$$

where $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\mathcal{M}_4$ *are smooth functions of time and*

$$\mathcal{M}(t)=(\frac{\sqrt{1+\kappa_g^2(t)}}{\sin\mathcal{E}(t)}-\cos\mathcal{E}(t))$$

and

$$\mathcal{V}(t)=\sqrt{1+\kappa_g^2(t)}-\frac{1}{2}\sin 2\mathcal{E}(t).$$

Proof. Assume that $\mathcal{P}(\sigma,u,t)$ be a one-parameter family of the $\mathcal{S}-\mathbf{s}$ surface of a unit speed non-geodesic biharmonic $\mathcal{S}$-curve.

From our assumption, we get the following equation

$$\mathbf{t}=\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]\mathbf{e}_1+\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]\mathbf{e}_2+\cos\mathcal{E}(t)\mathbf{e}_3. \quad (17)$$

where $\mathcal{M},\mathcal{M}_1$ are smooth functions of time.

Obviously, we also obtain

$$\mathbf{s}=\frac{1}{\kappa_g(t)}[\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)](\mathcal{M}(t)+\cos\mathcal{E}(t))-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_2(t)]\mathbf{e}_1$$
$$+\frac{1}{\kappa_g(t)}[-\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)](\mathcal{M}(t)+\cos\mathcal{E}(t))+\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_3]\mathbf{e}_2 \quad (18)$$
$$+\frac{1}{\kappa_g}[\cos\mathcal{E}\sigma-\frac{\mathcal{V}\sigma+\mathcal{M}_1}{2\mathcal{V}^2}\sin^4\mathcal{E}-\frac{\sin 2[\mathcal{M}\sigma+\mathcal{M}_1]}{4\mathcal{V}^2}\sin^4\partial\mathcal{E}-[\frac{\sin^2\mathcal{E}}{\mathcal{V}}\sin[\mathcal{M}\sigma+\mathcal{M}_1]+\mathcal{M}_3][-\frac{\sin^2\mathcal{E}}{\mathcal{V}}\cos[\mathcal{M}\sigma+\mathcal{M}_1]+\mathcal{M}_2]$$
$$+\frac{\mathcal{M}_2}{\mathcal{V}}\sin^3\mathcal{E}\sin[\mathcal{M}\sigma+\mathcal{M}_1]+\mathcal{M}_4]\mathbf{e}_3,$$

where $\mathcal{M},\mathcal{M}_1$ are smooth functions of time and

$$\mathcal{M}(t)=(\frac{\sqrt{1+\kappa_g^2(t)}}{\sin\mathcal{E}(t)}-\cos\mathcal{E}(t))\text{ and }\mathcal{V}(t)=\sqrt{1+\kappa_g^2(t)}-\frac{1}{2}\sin 2\mathcal{E}(t).$$

Furthermore, we have the natural frame $\{(\mathcal{P}^{\mathcal{S}})_\sigma,(\mathcal{P}^{\mathcal{S}})_u\}$ given by

$$(\mathcal{P}^{\mathcal{S}})_\sigma=(1-\kappa_g(t))\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]\mathbf{e}_1+(1-\kappa_g(t))\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]\mathbf{e}_2+(1-\kappa_g(t))\cos\mathcal{E}(t)\mathbf{e}_3$$

and

$$(\mathcal{P}^{\mathcal{S}})_u=\frac{1}{\kappa_g(t)}[\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)](\mathcal{M}(t)+\cos\mathcal{E}(t))-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_2(t)]\mathbf{e}_1$$
$$+\frac{1}{\kappa_g(t)}[-\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)](\mathcal{M}(t)+\cos\mathcal{E}(t))+\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_3]\mathbf{e}_2 \quad (19)$$
$$+\frac{1}{\kappa_g(t)}[\cos\mathcal{E}(t)\sigma-\frac{\mathcal{V}(t)\sigma+\mathcal{M}_1(t)}{2\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)-\frac{\sin 2[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]}{4\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)$$
$$-[\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_3(t)][-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]$$
$$+\mathcal{M}_2(t)]+\frac{\mathcal{M}_2(t)}{\mathcal{V}(t)}\sin^3\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_4(t)]\mathbf{e}_3$$

The components of the first fundamental form are

$$\frac{\partial\mathbf{E}}{\partial t}=\frac{\partial}{\partial t}g((\mathcal{P}^{\mathcal{S}})_\sigma,(\mathcal{P}^{\mathcal{S}})_\sigma)=\frac{\partial}{\partial t}[(1-\kappa_g(t))\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]]^2+\frac{\partial}{\partial t}[(1-\kappa_g(t))\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]]^2+\frac{\partial}{\partial t}[(1-\kappa_g(t))\cos\mathcal{E}(t)]^2,$$

$$\frac{\partial\mathbf{F}}{\partial t}=0,$$

$$\frac{\partial\mathbf{G}}{\partial t}=\frac{\partial}{\partial t}g((\mathcal{P}^{\mathcal{S}})_u,(\mathcal{P}^{\mathcal{S}})_u)=\frac{\partial}{\partial t}[\frac{1}{\kappa_g(t)}[\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)](\mathcal{M}(t)+\cos\mathcal{E}(t))-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_2(t)]]^2$$
$$+\frac{\partial}{\partial t}[\frac{1}{\kappa_g(t)}[-\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)](\mathcal{M}(t)+\cos\mathcal{E}(t))$$

$$+\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_3]]^2 \quad (20)$$

$$+\frac{\partial}{\partial t}[\frac{1}{\kappa_g(t)}[\cos\mathcal{E}(t)\sigma-\frac{\mathcal{V}(t)\sigma+\mathcal{M}_1(t)}{2\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)-\frac{\sin 2[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]}{4\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)$$

$$-[\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_3(t)][-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]$$

$$+\mathcal{M}_2(t)]+\frac{\mathcal{M}_2(t)}{\mathcal{V}(t)}\sin^3\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_4]]^2.$$

Hence, $\frac{\partial\mathcal{P}^{\mathcal{S}}}{\partial t}$ is inextensible if and only if Eq. (16) is satisfied. This concludes the proof of theorem.

Theorem 4.4. *Let $\mathcal{P}^{\mathcal{S}}$ be one-parameter family of the $\mathcal{S}-\mathbf{s}$ surface of a unit speed non-geodesic biharmonic $\mathcal{S}$-curve. Then, the parametric equations of this family are*

$$x_{\mathcal{P}^{\mathcal{S}}}(\sigma,u,t)=-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+u[\frac{\sin\mathcal{E}(t)}{\kappa_g(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]$$

$$(\mathcal{M}(t)+\cos\mathcal{E}(t))-\frac{\sin^2\mathcal{E}}{\kappa_g\mathcal{V}}\cos[\mathcal{M}\sigma+\mathcal{M}_1]+\mathcal{M}_2]+\mathcal{M}_2,$$

$$y_{\mathcal{P}^{\mathcal{S}}}(\sigma,u,t)=\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+u[-\frac{\sin\mathcal{E}(t)}{\kappa_g(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)] \quad (21)$$

$$(\mathcal{M}(t)+\cos\mathcal{E}(t))+\frac{\sin^2\mathcal{E}(t)}{\kappa_g(t)\mathcal{V}(t)}\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_3(t)]+\mathcal{M}_3(t),$$

$$z_{\mathcal{P}^{\mathcal{S}}}(\sigma,u,t)=\cos\mathcal{E}(t)\sigma-\frac{\mathcal{V}(t)\sigma+\mathcal{M}_1(t)}{2\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)-\frac{\sin 2[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]}{4\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)$$

$$+\frac{\mathcal{M}_2(t)}{\mathcal{V}(t)}\sin^3\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+u[\cos\mathcal{E}(t)+\sin\mathcal{E}(t)(-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_2(t))\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]]$$

$$+u[-\frac{\sin^2\mathcal{E}(t)}{2\kappa_g(t)}(\mathcal{M}(t)+\cos\mathcal{E}(t))^2\sin 2[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]$$

$$+\frac{\cos\mathcal{E}(t)}{\kappa_g(t)}\sigma-\frac{\mathcal{V}(t)\sigma+\mathcal{M}_1(t)}{2\kappa_g(t)\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)-\frac{\sin 2[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]}{4\kappa_g(t)\mathcal{V}^2(t)}\sin^4\mathcal{E}(t)$$

$$+\frac{\mathcal{M}_2(t)}{\kappa_g(t)\mathcal{V}(t)}\sin^3\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]+\mathcal{M}_4(t)]+\mathcal{M}_4(t),$$

where $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\mathcal{M}_4$ are smooth functions of time and

$$\mathcal{M}(t)=(\frac{\sqrt{1+\kappa_g^2(t)}}{\sin\mathcal{E}(t)}-\cos\mathcal{E}(t)) \quad \textit{and}$$

$$\mathcal{V}(t)=\sqrt{1+\kappa_g^2(t)}-\frac{1}{2}\sin 2\mathcal{E}(t)\cdot$$

Proof. By the Sabban formula, we have the following equation

$$\mathbf{t}=\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]\mathbf{e}_1+\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]\mathbf{e}_2+\cos\mathcal{E}\mathbf{e}_3. \quad (22)$$

Using (2.1) in (4.10), we obtain

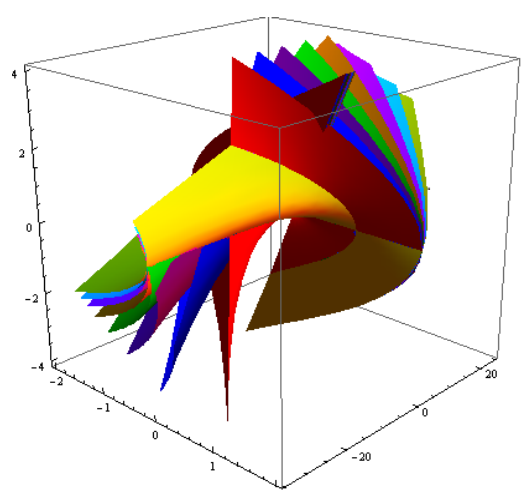
$$t=(\sin\mathcal{E}(t)\sin[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)],\sin\mathcal{E}(t)\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)],$$

$$\cos\mathcal{E}(t)+\sin\mathcal{E}(t)(-\frac{\sin^2\mathcal{E}(t)}{\mathcal{V}(t)}\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)] \quad (23)$$
$$+\mathcal{M}_2(t))\cos[\mathcal{M}(t)\sigma+\mathcal{M}_1(t)]).$$

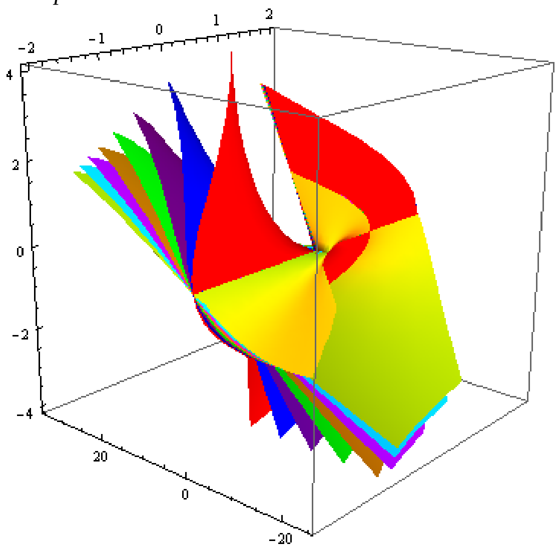
where $\mathcal{M}_1,\mathcal{M}_2$ are smooth functions of time.

Consequently, the parametric equations of $\mathcal{P}^{\mathcal{S}}$ can be found from (13), (23). This concludes the proof of Theorem.

We can use Mathematica in above theorem, yields



(a)



(b)

FIGURE 1. (a) and (b) The equation (21) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Yellow, Green at the time t = 1, t = 1.2, t = 1.4, t = 1.6, t = 1.8, t = 2, t = 2.2, t = 2.4, respectively.

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