Determining the maximum or minimum impedance of a special parallel *RLC* circuit without calculus

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Abstract

We show that the maximum or minimum impedance of a special parallel *RLC* studied in a previous paper can be found analytically, without using calculus. In fact, we show that the maximum or minimum value occurs when the driving frequency ω is equal to $\omega_o = 1/\sqrt{LC}$, a fact that was determined graphically in that previous paper. Furthermore, we show that either the maximum or minimum value is given by $R\left(\rho^2+1\right)/\left(\rho^2+3\right)$, where $\rho=\sqrt{L/C}/R.$ Also, for $\rho^2 \ll 3,$ the minimum impedance $Z_{\min} \approx R\left(1+2\rho^2/3\right)/3 \approx R/3,$ whereas for $\rho^2 \gg 3,$ the maximum impedance $Z_{\max} \approx R\left(1-2/\rho^2\right) \approx R.$

Keywords: Parallel *RLC* circuit, maximum or minimum impedance, maximum or minimum without calculus.

Resumen

Demostramos que la impedancia máxima o mínima de un circuito RLC en paralelo estudiado en una publicación reciente puede ser derivada analíticamente sin usar cálculo. De hecho, mostramos que el valor máximo o mínimo ocurre cuando la frecuencia de conducción ω es igual a $\omega_o = 1/\sqrt{LC}$, un hecho que fue determinado en forma gráfica en dicha publicación. Mostramos además que el valor máximo o mínimo es $R\left(\rho^2+1\right)/\left(\rho^2+3\right)$, donde $\rho=\sqrt{L/C}/R.$ Para $\rho^2 \ll 3,$ la impedancia mínima es $Z_{\min} \approx R\left(1+2\rho^2/3\right)/3 \approx R/3,$ mientras que para $\rho^2 \gg 3,$ la impedancia máxima es $Z_{\max} \approx R\left(1-2/\rho^2\right) \approx R.$

Palabras Clave: Circuito *RLC* en paralelo, impedancia máxima o mínima, máximo o mínimo sin cálculo.

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I. INTRODUCTION

Ma *et al*., studied the interesting parallel *RLC* circuit of Fig. 1 in [1], for two specific cases: (i) $R_1 = R_2 = 0$ and $R_3 = R$ and (ii) $R_1 = R_2 = R_3 = R.$

They showed how the impedance Z as seen by the source varied with the angular frequency ω of the source. In fact, they plotted the normalized impedance Z/R as a function of the normalized angular frequency $\gamma = \omega/\omega_o$, where $\omega_o = 1/\sqrt{LC}$ for various values of the dimensionless parameter given by $\rho=\sqrt{L/C}/R.$ For case (i), they showed *graphically* that when $\omega=\omega_o \triangleq 1/\sqrt{LC},$ the impedance magnitude Z is a maximum value equal to R.

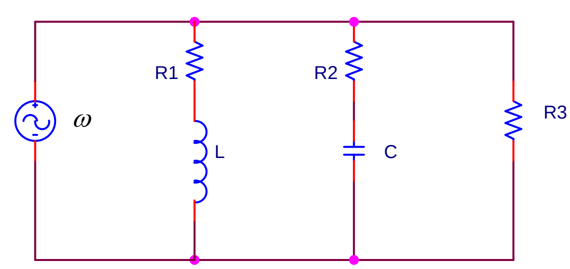

FIGURE 1. Schematic diagram of the parallel *RLC* circuit studied in [1], [2] and [3].

This result was apparently unexpected, as the authors state: "It is surprising to see that regardless of the ρ values,

$\left|Ze^{j\phi}/R\right|$ reaches to 1 (sic) when $\gamma=1$." However, Cartwright and Kaminsky [2] showed that this result could be predicted mathematically without the use of calculus.

For case (ii), the authors of [1] also showed *graphically* that for $\omega=\omega_o \triangleq 1/\sqrt{LC}$, the impedance magnitude Z is a maximum or a minimum value, except in the case of $\rho=1$, when the impedance is independent of frequency. In fact, a plot of the relationship between the normalized impedance Z/R and the normalized angular frequency $\gamma=\omega/\omega_o$ for various ρ values is given in Fig. 4 of [1] and a similar graph is given in Fig. 2 in Section II below. From these graphs, it does appear that the maximum or minimum Z occurs when $\gamma=1$, as noted in [1]. However, it would be rewarding to show analytically that this is indeed the case. In fact, this is the purpose of this paper, *i.e.*, to show mathematically that the maximum or minimum impedance does occur at $\gamma=1$. Furthermore, we do this algebraically, *i.e.*, without calculus. The maximum or minimum impedance value will then be determined by simply substituting $\gamma=1$ into the equation relating normalized impedance and γ, *i.e.*, Eq. (4) below.

For completeness, we also mention that Cartwright *et al.* [3] recently studied the circuit of Fig. 1 in detail for $R_1=R, R_2=0$ and $R_3=\infty$. In fact, the maximum impedance and the frequency at which it occurs ($\gamma\neq 1$) were derived for this case, without using calculus.

II. DERIVATION OF *Z* FOR THE CIRCUIT OF FIG. 1.

As given in [1], the complex impedance $\hat{Z}=Ze^{i\phi}$ for the circuit of Fig. 1, case (ii) is given by

$$\frac{\hat{Z}}{R}=\frac{Ze^{i\phi}}{R}=\left(1+\frac{1}{1+i\omega L/R}+\frac{1}{1-\dfrac{i}{\omega RC}}\right)^{-1}. \qquad (1)$$

Eq. (1) can also be written in terms of the dimensionless quantities γ and ρ. Indeed, as given in [1],

$$\frac{\hat{Z}}{R}=\frac{Ze^{i\phi}}{R}=\left(1+\frac{1}{1+i\gamma\rho}+\frac{1}{1-i\rho/\gamma}\right)^{-1}. \qquad (2)$$

Using straightforward mathematical operations, Eq. (2) becomes

$$\frac{Ze^{i\phi}}{R}=\frac{\gamma\left(\rho^2+1\right)+i\rho\left(\gamma^2-1\right)}{\gamma\left(\rho^2+3\right)+i2\rho\left(\gamma^2-1\right)}. \qquad (3)$$

Hence, it follows that

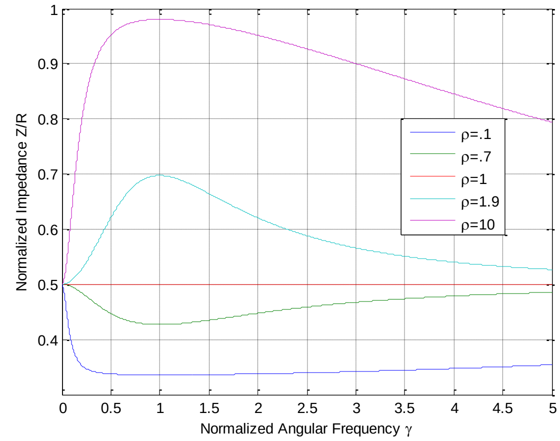

FIGURE 2. Relationship between the normalized impedance magnitude Z/R and the normalized angular frequency $\gamma=\omega/\omega_o$ for various $\rho=\sqrt{L/C}/R$ values.

$$\frac{Z}{R}=\left|\frac{Ze^{i\phi}}{R}\right|=\sqrt{\frac{\gamma^2\left(\rho^2+1\right)^2+\rho^2\left(\gamma^2-1\right)^2}{\gamma^2\left(\rho^2+3\right)^2+4\rho^2\left(\gamma^2-1\right)^2}}. \qquad (4)$$

Note that Eq. (4) can be used to generate the curves in Fig. 2. However, neither Eq. (3) nor Eq. (4) is given in [1], so it is unclear what method the authors of [1] used to produce their curves, although they likely simply used Matlab to compute the magnitude of their Eq. (6).

Interestingly, when $\rho=1$, Eq. (4) becomes $Z/R=1/2$, *i.e.*, the impedance is no longer a function of the radian frequency ω, in agreement with Fig. 2.

Furthermore, when $\gamma=1$, Eq. (4) gives

$$\frac{Z}{R}=\frac{\rho^2+1}{\rho^2+3}. \qquad (5)$$

So, if we can show analytically that the maximum or minimum occurs at $\gamma=1$, then Eq. (5) would give that minimum or maximum Z value.

In order to find the minimum or maximum of Eq. (4) without calculus, it will be necessary to write Eq. (4) in terms of the quotient of polynomials in γ. Hence, to accomplish this, notice that Eq. (4) can be rewritten as

$$\frac{Z}{R}=\sqrt{\frac{\gamma^2\left(\dfrac{\rho^2+1}{\rho}\right)^2+\left(\gamma^2-1\right)^2}{\gamma^2\left(\dfrac{\rho^2+3}{\rho}\right)^2+4\left(\gamma^2-1\right)^2}}. \qquad (6)$$

Furthermore,

$$\frac{Z}{R}=\frac{1}{2}\sqrt{\frac{\gamma^4+(A-2)\gamma^2+1}{\gamma^4+(B-2)\gamma^2+1}}, \qquad (7)$$

where $A=\left(\frac{\rho^2+1}{\rho}\right)^2$ and $B=\left(\frac{\rho^2+3}{2\rho}\right)^2$.

III. NON-CALCULUS DERIVATION OF THE MAXIMUM OR MINIMUM Z FOR THE CIRCUIT OF FIG. 1

Now that a mathematical expression has been determined in Eq. (7) for the normalized impedance magnitude, we can show how its maximum or minimum value can be obtained.

We note that Eq. (7) can be rewritten as

$$\begin{aligned}\frac{Z}{R}&=\frac{1}{2}\sqrt{\frac{\gamma^4+(B-2)\gamma^2+1+(A-B)\gamma^2}{\gamma^4+(B-2)\gamma^2+1}}\\&=\frac{1}{2}\sqrt{1+\frac{(A-B)\gamma^2}{\gamma^4+(B-2)\gamma^2+1}}\\&=\frac{1}{2}\sqrt{1+\frac{A-B}{\gamma^2+\gamma^{-2}-2+B}}\\&=\frac{1}{2}\sqrt{1+\frac{A-B}{\left(\gamma^1-\gamma^{-1}\right)^2+B}}.\end{aligned} \qquad (8)$$

There are now three cases to consider: $A=B, A>B$ and $A<B$.

A. Impedance when *A*=*B*

When $A=B$, $\rho=1$ and Eq. (8) becomes $Z/R=1/2$ as mentioned earlier.

B. Maximum Impedance when *A*>*B*

When $A>B$, it is easily shown that $\rho>1$. Hence, as is evident from Fig. 2, there is a maximum value of *Z*. Furthermore, from Eq. (8), it is clear that the impedance is maximized if $\frac{(A-B)}{\left(\gamma^1-\gamma^{-1}\right)^2+B}$ is maximized. As $A-B$ is positive, $\frac{(A-B)}{\left(\gamma^1-\gamma^{-1}\right)^2+B}$ is maximized when $\left(\gamma^1-\gamma^{-1}\right)^2$ is minimized, *i.e.*, when $\gamma=1$.

Hence, we have shown analytically that the impedance is maximized when $\gamma=1$ for $\rho>1$; therefore, its maximum value is given by Eq. (5). Alternatively, substituting $\gamma=1$ into Eq. (8) gives the maximum impedance as

$$\frac{Z_{\max}}{R}=\frac{1}{2}\sqrt{\frac{A}{B}}=\frac{\rho^2+1}{\rho^2+3}=\frac{1+\frac{1}{\rho^2}}{1+\frac{3}{\rho^2}}=\left(1+\frac{1}{\rho^2}\right)\left(1+\frac{3}{\rho^2}\right)^{-1}. \qquad (9)$$

Recall that for small x, $(1+x)^{-1}\approx 1-x$ (see *e.g.*, [4]); therefore, for large ρ^2, *i.e.*, $\rho^2\gg 3$, $\left(1+\frac{3}{\rho^2}\right)^{-1}\approx 1-\frac{3}{\rho^2}$. Hence, Eq. (9) becomes (ignoring powers higher than second order),

$$\frac{Z_{\max}}{R}\approx 1-\frac{2}{\rho^2}=1-2\frac{RC}{\frac{L}{R}}=1-2\frac{\tau_C}{\tau_L}, \qquad (10)$$

where $\tau_L=L/R$ and $\tau_C=RC$ are time-constants of the circuit.

Furthermore, for large ρ^2, $\frac{2}{\rho^2}$ is small compared to one: therefore, Eq. (10) reduces to $Z_{\max}\approx R$.

C. Minimum Impedance when *A*<*B*

When $A<B$, it is easily shown that $\rho<1$. Hence, as is evident from Fig. 2, there is a minimum value of *Z*.

We can rewrite Eq. (8) as

$$\frac{Z}{R}=\frac{1}{2}\sqrt{1-\frac{B-A}{\left(\gamma^1-\gamma^{-1}\right)^2+B}}. \qquad (11)$$

Note that $B-A$ is positive. Furthermore, from Eq. (11), it is clear that the impedance is minimized if $\frac{B-A}{\left(\gamma^1-\gamma^{-1}\right)^2+B}$ is maximized, which occurs when $\left(\gamma^1-\gamma^{-1}\right)^2$ is minimized, *i.e.*, when $\gamma=1$.

Hence, we have shown analytically that the impedance is minimized when $\gamma=1$ for $\rho<1$; therefore, its minimum value is given by Eq. (5). Alternatively, substituting $\gamma=1$ into Eq. (11) gives the minimum impedance as

$$\begin{aligned}\frac{Z_{\min}}{R}&=\frac{1}{2}\sqrt{\frac{A}{B}}=\frac{\rho^2+1}{\rho^2+3}=\frac{\rho^2+1}{3\left(1+\frac{\rho^2}{3}\right)}\\&=\frac{1}{3}\left(\rho^2+1\right)\left(1+\frac{\rho^2}{3}\right)^{-1}.\end{aligned} \qquad (12)$$

For small ρ^2, *i.e.*, $\rho^2/3 \ll 1$, or $\rho^2 \ll 3$, $\left(1+\frac{\rho^2}{3}\right)^{-1} \approx 1-\frac{\rho^2}{3}$.

Hence, Eq. (12) becomes

$$\begin{aligned}\frac{Z_{\min}}{R} &\approx \frac{1}{3}\left(\rho^2+1\right)\left(1-\frac{1}{3}\rho^2\right) \approx \frac{1}{3}\left(1+\frac{2}{3}\rho^2\right) \\ &= \frac{1}{3}\left(1+\frac{2}{3}\frac{L}{R}\frac{1}{RC}\right) \\ &= \frac{1}{3}\left(1+\frac{2}{3}\frac{\tau_L}{\tau_C}\right).\end{aligned} \tag{13}$$

For small ρ^2, $\frac{2}{3}\rho^2$ is small compared to one: therefore, Eq. (13) reduces to $Z_{\min} \approx R/3$. Again, in deriving Eq. (13), we ignored powers higher than second order.

IV. CONCLUSIONS

We have shown that the maximum or minimum impedance of the parallel circuit of Fig. 1 can be determined without calculus. In fact, we have determined that the maximum or minimum impedance is given by Eq. (5), *i.e.*, $\frac{Z}{R} = \frac{\rho^2+1}{\rho^2+3}$.

Furthermore, we have shown that for $\rho^2 \ll 3$, the minimum impedance $Z_{\min} \approx R\left(1+2\rho^2/3\right)/3 \approx R/3$, whereas for $\rho^2 \gg 3$, the maximum impedance $Z_{\max} \approx R\left(1-2/\rho^2\right) \approx R$.

Finally, we would like to point out that these theoretical results can be verified with PSpice simulation as was done in [2]. However, the details are not that different from what was done in [2] and hence the PSpice simulation is not reported here.

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