Thomas rotation and Mocanu paradox – not at all paradoxical

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Abstract

Non-commutativity of the Einstein velocity addition, in case of non-collinear velocities, seemingly gives rise to a conflict with reciprocity principle. However, Thomas rotation comes at a rescue and the paradox is avoided. It is shown that such a resolution of the so called Mocanu paradox is completely natural from the point of view of basic premises of special relativity.

Keywords: Physics Education, Special relativity, Thomas rotation, Mocanu paradox

Resumen

La no-conmutatividad de la suma de la velocidad de Einstein, en el caso de las velocidades no-alineadas, al parecer da lugar a un conflicto con el principio de reciprocidad. Sin embargo, la rotación de Thomas llega como un rescate y la paradoja se puede evitar. Se demuestra que dicha resolución de la llamada paradoja de Mocanu es completamente natural desde el punto de vista de las premisas básicas de la relatividad especial.

Palabras clave: Enseñanza de la Física, Relatividad especial, rotación de Thomas, paradoja de Mocanu.

I. INTRODUCTION

It should be clear, after a hundred years of development of special relativity, that to search a logical contradiction or paradoxes in it is the same as to search a logical inconsistency in non-Euclidean geometry (in fact, special relativity *is* a kind of non-Euclidean geometry – the Minkowski geometry of space-time). Surprisingly, however, such efforts have never been abandoned. Some "paradoxes" are helpful nevertheless because their resolution reveals the roots of our confusion and, therefore, enhances our comprehension of special relativity.

The Mocanu paradox [1, 2, 3] is an interesting paradox of this kind whose resolution makes clear some our misconceptions about space and time, deeply rooted in Newtonian intuition, which are notoriously hard to eliminate in physics students even after years of study of modern physics.

Although the resolution of this "paradox" is already available in the literature (see [3, 4, 5, 6]), "their arguments and mathematical formulas in terms of coordinates do not give an evident physical explanation of the paradox, though it became clear that the paradox was related somehow to the Thomas rotation" [6].

It is the aim of this article to demonstrate by elementary means that there is nothing especially paradoxical about the Thomas rotation as far as it is considered with regard to the Mocanu paradox. To emphasize the physical concepts involved, rather than mathematical formalism, we consider not the most general case of the Mocanu paradox. However, the special case considered already involves all necessary ingredients.

II. THE MOCANU PARADOX

Suppose a reference frame S' moves with the velocity v with respect to the frame "at rest", S, along its x-axis, and a frame S'' moves with the velocity v' with respect to the frame S' along its y'-axis. It is assumed that the corresponding axes of the frames S and S' are parallel to each other, as do axes of the frames S' and S''. Then the velocity $\vec{u}$ of S'' relative to S is given by the relativistic velocity addition law

$$u_x = \frac{v'_x + v}{1 + \frac{v'_x v}{c^2}} = v, \quad u_y = \frac{v'_y}{\gamma\left(1 + \frac{v'_x v}{c^2}\right)} = \frac{v'}{\gamma},$$

$$u_z = \frac{v'_z}{\gamma\left(1 + \frac{v'_x v}{c^2}\right)} = 0,$$

(1)

where γ is the Lorentz factor corresponding to the velocity v

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1 - \beta^2}}.$$

According to the reciprocity principle [7], if S' moves relative to S with velocity $\vec{v}$, then S moves relative to S' with velocity $-\vec{v}$. Therefore, in the frame S'', the frame S' moves along the y'' axis with the velocity $-v'$, while in the frame S', the frame S moves along the x' axis with the velocity $-v$. Compared to the previous situation, the roles of the x and y axes are interchanged, as are the roles of v and v' (with additional change of sign). Therefore, the velocity addition formula gives the velocity $\vec{u}'$ of the frame S relative to S''

$$u'_x = -\frac{v}{\gamma'}, \quad u'_y = -v', \quad u'_z = 0,$$

(2)

where γ' corresponds to the velocity v'.

Of course, it is possible to obtain all this by using the general formula for relativistic addition of non-collinear velocities [3]

$$\vec{u} = \vec{v} \oplus \vec{v}' = \frac{\vec{v} + \vec{v}'}{1 + \frac{\vec{v} \cdot \vec{v}'}{c^2}} + \frac{1}{c^2} \frac{\gamma}{\gamma + 1} \frac{\vec{v} \times (\vec{v} \times \vec{v}')}{1 + \frac{\vec{v} \cdot \vec{v}'}{c^2}},$$

(3)

from which a non-commutativity of this addition is clearly seen, but for our purposes even simpler particular case of this formula for the velocity $\vec{v}$ collinear to the x-axis, (1), suffices if carefully used.

According to the reciprocity principle, the velocity of S'' relative to S should be $-\vec{u}' = \vec{v}' \oplus \vec{v}$, but it clearly does not equal to $\vec{u} = \vec{v} \oplus \vec{v}'$. And this constitutes the content of the Mocanu paradox: what is the correct velocity of S'' relative to S, $\vec{v} \oplus \vec{v}'$ or $\vec{v}' \oplus \vec{v}$, and how we can account for the reciprocity principle in this case?

We can discard a possibility that the reciprocity principle is violated from the very beginning. In fact, it is possible and even preferable to base special relativity on this intuitively evident principle, instead of highly counter-intuitive second postulate (see [8] and references therein).

III. RESOLUTION OF THE MOCANU PARADOX

The key idea in resolution of the Mocanu paradox is the realization of the fact that space in special relativity is in fact more relative than space in the non-relativistic physics [6], although this can hardly be guessed by merely comparing the Galilean transformation $x' = x - vt$, which describes relativity of space for non-relativistic observers, to its relativistic counterpart $x' = \gamma(x - vt)$. In words of Minkowski, "space by itself, and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality" [9].

The vectors $\vec{v} \oplus \vec{v}'$ and $\vec{v}' \oplus \vec{v}$ are defined in different reference frames S and S'', and, therefore, in different spaces. It makes no sense to compare them unless the axes of S and S'' are made parallel in some well defined way.

Axes of the S and S', as well as axes of the S' and S'' frames are assumed to be parallel, as mentioned above. What conclusion we can draw then about the mutual orientation of the S and S'' frames axes?

In the frame S', the x'' axis is given by the equation (we will drop z-coordinate as it is irrelevant in our planar case)

$$y' = v't'.$$

Then, according to Lorentz transformations, we conclude that in the frame S the x'' axis is given by the equation

$$y = v'\gamma\left(t - \frac{v}{c^2}x\right).$$

Therefore, from the point of view of S, the x'' axis is inclined clockwise relative to the x axis by an angle α so that

$$\tan\alpha = \beta\beta'\gamma.$$

(4)

There is nothing paradoxical in this change of inclination. At least nothing more paradoxical than the lack of absolute simultaneity from which it stems. Note that such a change of inclination is used to resolve some pole-and-barn type paradoxes [10, 11].

Analogously, y'' axis is given in the frame S' by the equation $x' = 0$, which in the frame S transforms into

$$\gamma(x - vt) = 0.$$

Therefore, y'' axis is given in the frame S by the equation $x = vt$ and, consequently, remains parallel to the y axis. Fig. 1 summarizes the orientations of the x'' and y'' axes as seen by an observer in the S reference frame.
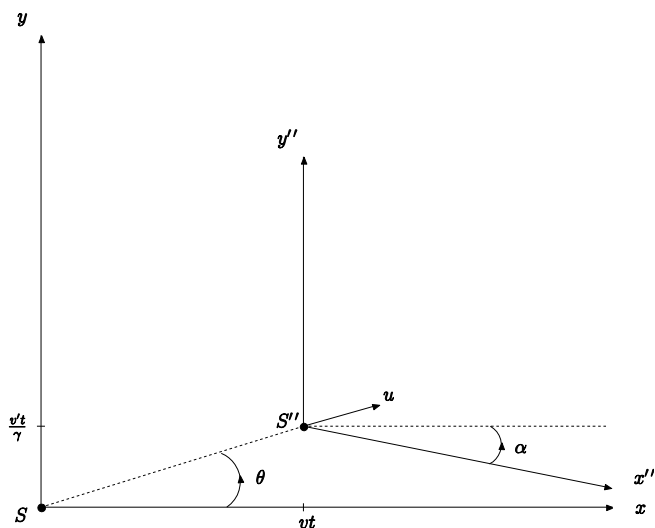
FIGURE 1. Orientations of the S'' axes as perceived in the frame S.

We need some refinement here. Because of the finite speed of light, we should distinguish between what Rindler calls [12] world-picture and world-map. World-picture is what an observer actually sees at any given moment of time, a snapshot which records distant objects at different moments of the past. World-map, on the contrary, is the set of events that the observer considers to have occurred in the world at that instant of time. Special relativity operates with world-maps, Lorentz transformation being an instrument which relates two world-maps of different inertial frames. Therefore, when we speak rather loosely about what an observer sees or perceives, actually we have in mind the world-map of this observer. With this caveat, let us continue and find how the situation described by Fig. 1 is transformed in the frame S''.
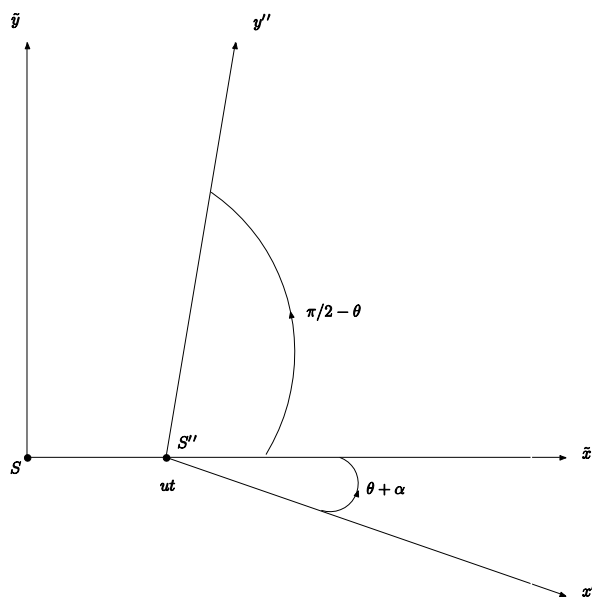

FIGURE 2. Orientations of the S'' axes as perceived in the frame S relative to the $\tilde{x}$ and $\tilde{y}$ axes.

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First of all, let us introduce another set of axes $\tilde{x}, \tilde{y}$ and $\tilde{x}'', \tilde{y}''$, so that $\tilde{x}$ and $\tilde{x}''$ are parallel to $\vec{u}$ and, therefore, S and S'' equipped with these axes are in a standard configuration. In these new axes, Fig. 1 is changed into Fig. 2.

But

$$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{\dfrac{\beta'}{\beta\gamma} + \beta\beta'\gamma}{1 - \beta'^2} = \frac{\gamma\beta'\gamma'^2}{\beta},$$

and

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta = \frac{\beta\gamma}{\beta'}.$$

Therefore, the equation which defines x'' axis in the frame S looks like

$$\tilde{y} = -\frac{\gamma\beta'\gamma'^2}{\beta}(\tilde{x} - ut), \tag{5}$$

while the equation for the y'' axis is

$$\tilde{y} = \frac{\beta\gamma}{\beta'}(\tilde{x} - ut). \tag{6}$$

Let us apply now the Lorentz transformation

$$\tilde{x} = \gamma_u(\tilde{x}'' + ut''), \ \ t = \gamma_u\left(t'' + \frac{u}{c^2}\tilde{x}''\right), \ \ \tilde{y} = \tilde{y}'',$$

to change world-map from S to S''. As a result, we get from (5)

$$\tilde{y}'' = -\frac{\gamma\beta'\gamma'^2}{\beta}\gamma_u(1 - \beta_u^2)\tilde{x}'' =$$
$$-\frac{\gamma\beta'\gamma'^2}{\beta\gamma_u}\tilde{x}'' = -\frac{\beta'\gamma'}{\beta}\tilde{x}'', \tag{7}$$

where at the last step we have used

$$\gamma_u = \gamma_{\vec{v}\oplus\vec{v}'} = \gamma\gamma'\left(1 + \frac{\vec{v}\cdot\vec{v}'}{c^2}\right) = \gamma\gamma'.$$

Analogously, (6) transforms into

$$\tilde{y}'' = \frac{\beta\gamma}{\beta'\gamma_u}\tilde{x}'' = \frac{\beta}{\beta'\gamma'}\tilde{x}''. \tag{8}$$

Equations (7) and (8) show that, from the point of view of an observer in the frame S'', x'' and y'' axes are inclined with respect to the $\tilde{x}''$ axis (and, hence, with respect to the

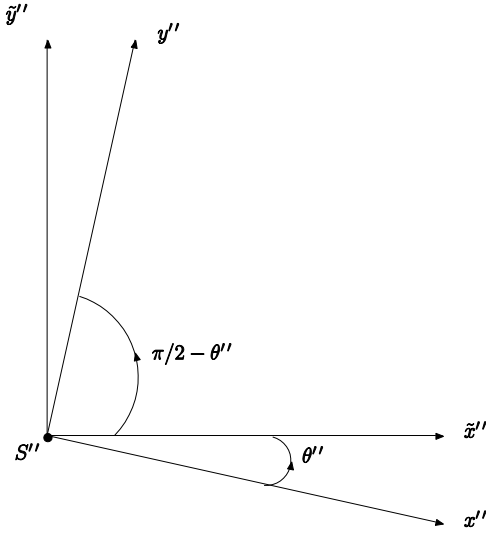line of relative motion) with angles $-\theta''$ and $\pi/2-\theta''$ respectively, as shown in Fig. 3.

FIGURE 3. Orientations of the S'' axes x'' and y'' as perceived in the frame S'' relative to the $\tilde{x}''$ and $\tilde{y}''$ axes.

At that

$$\tan\theta'' = \frac{\beta'\gamma'}{\beta}. \tag{9}$$

While naively one should expect the inclination angle θ'' to be the same angle θ by which the x axis is inclined with respect to the $\tilde{x}$ axis in the frame S, as it would be in the case of parallel x and x'' axes of non-relativistic situation, it is not, because

$$\tan\theta = \frac{\beta'}{\beta\gamma}. \tag{10}$$

As we see, although axes of the frames S and S', as well as S' and S'', were rendered parallel, the very same axes of the frames S and S'' turned out not to be parallel in any meaningful way. Space for relativistic observers are more relative than for non-relativistic observers and we should be very careful while interpreting the results of several consecutive non-collinear boosts.

The difference $\varepsilon = \theta'' - \theta$ is the notorious Thomas rotation and it provides a ready explanation of the Mocanu paradox: an observer in the frame S'' really perceives $-\bar{u}$ as the velocity of the frame S, in agreement with the reciprocity principle, but projects this vector of relative velocity onto x'' and y'' axes to get its components.

$$u'_x = -u\cos\theta'' = -\frac{u}{\sqrt{1+\tan^2\theta''}} = -\frac{v}{\gamma'},$$

$$u'_y = u'_x\tan\theta'' = -v',$$

where we have used

$$u = v\sqrt{1+\frac{\beta'^2}{\beta^2\gamma^2}} = v\sqrt{1+\frac{\beta'^2}{\beta^2}\left(1-\beta^2\right)} =$$

$$v\sqrt{\frac{1}{\gamma'^2}+\frac{\beta'^2}{\beta^2}} = \frac{v}{\gamma'}\sqrt{1+\tan^2\theta''}.$$

As we see, Thomas rotation resolves the Mocanu paradox and reconciles the reciprocity principle with the non-commutativity of relativistic velocity addition.

IV. CONCLUDING REMARKS

Thomas rotation and Thomas precession are often considered as obscure relativistic effects which have generated a huge, sometimes confusing literature [13]. Nevertheless, this phenomena "can be quite naturally introduced and investigated in the context of a typical introductory course on special relativity, in a way that is appropriate for, and completely accessible to, undergraduate students" [14]. I think the Mocanu paradox provides a very useful possibilities in this respect.

The resolution of the paradox presented in this article was essentially given by Ungar [3] years ago. I hope, however, that the above presentation is simpler and clarifies some confusion. For example, it is claimed in [3] that an observer in S sees the axes of S'' rotated relative to his own axes by a Thomas rotation angle ε. However, this is not correct. The observer in S "sees" what is depicted in Fig. 1. Thomas rotation angle, ε, emerges when we compare the orientation of S'' axes, as actually seen by an observer in S'', to the naive expectation of the observer in S what the observer in S'' should see if the transitivity of parallelism is assumed between different inertial reference frames.

Thomas rotation is very basic phenomenon in special relativity which follows quite naturally from its basic premises, as was demonstrated above. It is as basic as the time dilation and length contraction and is no more paradoxical than these well known effects of special relativity. Of course, this does not mean that it is trivial. It took years before "evidence that Einstein's addition is regulated by the Thomas precession has come to light, turning the notorious Thomas precession, previously considered the ugly duckling of special relativity theory, into the beautiful swan of gyrogroup and gyrovector space theory" [15]. At this more advanced level, you can enjoy also other non-Euclidean facets of relativistic velocity space [16], from which the geometrical meaning of Thomas rotation, first discovered by the famous French

mathematician Émile Borel long before Thomas found the precession effect [17], becomes evident.

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REFERENCES

[1] Mocanu, C. I., *Some difficulties within the framework of relativistic electrodynamics*, Arch. Elektrotech. **69**, 97–110 (1986).

[2] Mocanu, C. I., *On the relativistic velocity composition paradox and the Thomas rotation*, Found. Phys. Lett. **5**, 443–456 (1992).

[3] Ungar, A. A., *The relativistic velocity composition paradox and the Thomas rotation*, Found. Phys. **19**, 1385–1396 (1989).

[4] Ungar, A. A., *The Relativistic Composite-Velocity Reciprocity Principle*, Found. Phys. **30**, 331–342 (2000).

[5] Good, I. J., *Lorentz matrices: A review*, Int. J. Theor. Phys. **34**, 779–799 (1995).

[6] Matolcsi, T., Goher, A., *Spacetime without Reference Frames: An Application to the Velocity Addition Paradox*, Stud. Hist. Philos. Mod. Phys. **32**, 83–99 (2001).

[7] Berzi, V., Gorini, V., *Reciprocity principle and the lorentz transformations*, J. Math. Phys. **10**, 1518–1524 (1969).

[8] Silagadze, Z. K., *Relativity without tears*, Acta Phys. Polon. B **39**, 811-885 (2008).

[9] Minkowski, H., *Space and Time*, In: Lorentz, H. A., Einstein, A., Minkowski, H., Weyl, H., The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity, (Dover, New York, 1952), p. 75.

[10] Shaw, R., *Length Contraction Paradox*, Am. J. Phys. **30**, 72–72 (1962).

[11] Iyer, C., Prabhu, G. M., *Differing observations on the landing of the rod into the slot*, Am. J. Phys. **74**, 998–1001 (2006).

[12] Rindler, W., *Relativity: Special, General, and Cosmological*, (Oxford University Press, Oxford, 2001).

[13] Malykin, G. B., *From the history of physics: Thomas precession: Correct and incorrect solutions*, Phys. Usp. **49**, 837–853 (2006).

[14] Costella, J. P., McKellar, B. H. J., Rawlinson, A. A., Stephenson, G. J., *The Thomas rotation*, Am. J. Phys. **69**, 837-847 (2001).

[15] Ungar, A. A., Beyond the Einstein Addition Law and Its Gyroscopic Thomas Precession. The Theory of Gyrogroups and Gyrovectors Spaces (Fundamental Theories of Physics, Vol. 117), (Kluwer Acad. Publ., Dordrecht, 2001).

[16] Rhodes, J. A., Semon, M. D., *Relativistic velocity space, Wigner rotation and Thomas precession*, Am. J. Phys. **72**, 943-960 (2004).

[17] Walter, S., *The Non-Euclidean Style of Minkowskian Relativity*, In: Gray, J. (ed.): The Symbolic Universe, (Oxford University Press, Oxford, 1999), pp. 91-127.