# Another look at the projectile motion

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**Abstract**

This paper discusses the traditional problem of the projectile motion by alternative approaches. The usual assumption of neglecting air resistance is considered. We verify that the vector method based on the whole position vector gives significant pedagogical advantage when compared with the decomposition method alone. We also show that the approaches stressing the concept of linear momentum and the conservation of energy play an essential role. The methodology based on the notion of first integrals of motion gives an overview on this topic, showing the interplay and complementarity of the different descriptions.

**Keywords:** Kinematics, projectile motion.

**Resumen**

Este artículo discute el problema tradicional del movimiento de un proyectil en enfoques alternativos. El supuesto habitual de dejar de lado la resistencia del aire es considerado. Verificamos que el método vectorial basado en la posición total del vector da ventaja pedagógica significativa cuando se compara solo con el método de descomposición. También se muestra que los enfoques dan hincapié al concepto de cantidad de movimiento y la conservación de la energía juega un papel esencial. La metodología basada en la noción de las integrales primeras del movimiento ofrece una visión general sobre este tema, mostrando la interacción y la complementariedad de las diferentes descripciones.

**Palabras clave:** Cinemática, movimiento de proyectiles.

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## I. INTRODUCTION

Projectile motion is a topic that students of the scientific areas learn at secondary school. Even when this subject is revisited in the introductory courses of physics in the university, students are typically invited to memorize and apply a pre-established prescription. This methodology is based on the traditional decomposition into two independent motions. As a matter of fact, we can observe that the generality of the textbooks only consider this traditional approach that is followed by teachers in the classroom. At this level, it would be expected that students could be stimulated to apply new points of view to such a familiar problem.

Several articles and notes have discussed the problem of projectile motion in two-dimensions in the absence of air resistance [1, 2, 3, 4, 5, 6, 7]. Some of them discuss interesting aspects like the orthogonality of initial and final velocities for the maximizing parabolic trajectory [3, 4]. In general, authors combine vectorial and analytical methods to obtain the main results. An alternative description which includes the concepts of angular momentum and torque to solve some aspects of this problem has also been considered [8].

In the present paper the projectile problem is revisited. The usual assumption of constant acceleration of gravity $\vec{g}$ and absence of air resistance will be made. Besides the traditional approach, three complementary methods are presented. The most straightforward method takes advantage of the whole position vector, with calculus being kept to a minimum. We report this situation by addressing two exploratory examples that are typically used by teachers to introduce this subject. A next step applies impulse-momentum and kinetic-energy theorems, stressing momentum and energy variables. Finally, a first integrals method gives an overview of the projectile problem, allowing a natural link between dynamical equations and constants of motion, *i.e.*, quantities whose values do not change along the entire path of the particle [9]. So, the variety of descriptions and points of view allows to get a deep understanding of this subject, making it a theoretical tool with a didactic interest in physics or mathematics courses. To prove the reliability of the nonconventional methods as powerful alternatives for solving problems of projectile motion in two-dimensions, two illustrative problems are also included.

We remember that the existence of conserved quantities of an isolated mechanical system is associated with its invariance under the Galilean group. However, it is also possible to prove the existence of other constants of motion rather than the well known quantities arising from space-time symmetries as the laws of conservation of momentum and energy [10]. Some of the constants of motion can depend explicitly on time.

Independently of the method used, we will also show that the velocity vector plays a crucial role. Within the methodology addressing the law of conservation of energy, this is visible in the way the equation of the trajectory can be calculated. Here we suggest the condition of parallelism between the velocity vector and the tangent to the trajectory. So, this procedure also illustrates the application of simple mathematical tools to physics, a barrier that some students have difficulty to overcome. However, it is worthwhile showing that the problems suggested can be solved even with a simple high school level of algebra, which most students taking an intermediate physics course at the university are supposed to be familiar with.

## II. TRADITIONAL TREATMENT

For convenience and comparative purposes, we start with a summary of the traditional approach to the projectile motion. This motion has a constant acceleration $\vec{a}=\vec{g}$, so the position vector as a function of time is given by

$$\vec{r}=\vec{r}_0+\vec{v}_0\,t+\frac{1}{2}\vec{g}\,t^2, \tag{1}$$

where $\vec{r}_0$ and $\vec{v}_0$ are the initial values of position and velocity vectors, respectively.

Using this approach, let us address an illustrative example of projectile motion which corresponds to the simpler physical situation $(\vec{r}_0=0)$, being in general used by teachers to introduce this subject.

*Example 1. A particle is thrown upward at an angle $\alpha$ to the horizontal and with an initial speed $v_0$. Determine the time of flight of the particle, the maximum range and height, and the equation of the trajectory.*

The traditional methodology is based on the decomposition of the two-dimensional motion into the horizontal and vertical components, where the discussion of the velocity vector and its components (see Fig. 1) play a fundamental role.

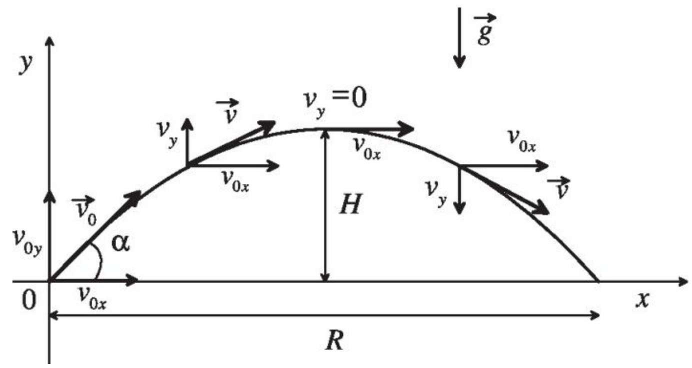


**FIGURE 1.** Parabolic trajectory of a projectile which leaves the origin with a velocity $\vec{v}_0$ at an angle $\alpha$. The components and the velocity vector in four point of the path are shown.

The relevant quantities of the projectile problem are summarized in the following expressions:

$$\text{Range, } x=v_0\cos\alpha\, t\,, \tag{2}$$

$$\text{Height, } y=v_0\sin\alpha\, t-\frac{1}{2}g\,t^2, \tag{3}$$

$$\text{Time of flight, } T=\frac{2v_0}{g}\sin\alpha\,, \tag{4}$$

$$\text{Maximum range, } R=\frac{v_0^2}{g}\sin 2\alpha\,, \tag{5}$$

$$\text{Maximum height, } H=\frac{v_0^2}{2g}\sin^2\alpha\,, \tag{6}$$

$$\text{Parabolic trajectory, } y=\tan\alpha\, x-\frac{g}{2v_0^2\cos^2\alpha}x^2. \tag{7}$$

## III. APPROACH BASED ON ELEMENTARY VECTOR ANALYSIS

After the presentation of the projectile motion, which starts with the equations of motion for each of the two dimensions, teachers can show the advantage and versatility of exploring geometrical and analytical aspects of the problem through whole vectors and its addition. The importance of using whole vectors in mechanics has been pointed out by Wheeler [11]. However, with few exceptions [12], this approach is absent from the literature and the classroom teaching. For *example 1*, the position vector of the projectile as a function of time follows directly from (1) with $\vec{r}_0=0$:

$$\vec{r}=\vec{v}_0\,t+\frac{1}{2}\vec{g}\,t^2. \quad (8)$$

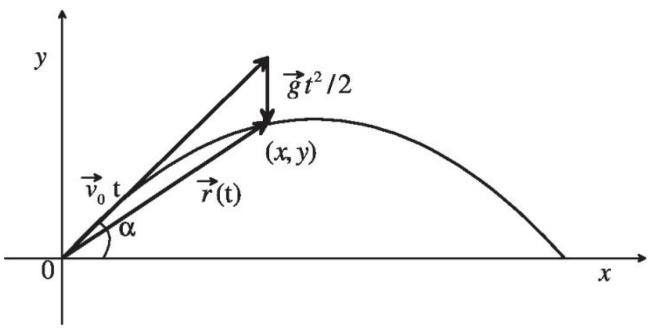


**FIGURE 2.** Position vector of a projectile with coordinates $(x, y)$ at the instant $t$ as the addition of two displacement vectors: the vector $\vec{v}_o\,t$ if the gravity were absent, and the vector $\vec{g}t^2/2$ arising from the gravity.

This expression is represented in Fig. 2 showing the addition of two terms: (1) $\vec{v}_0\,t$ is the displacement if no acceleration were present resulting from the initial velocity of the particle, and (2) $\frac{1}{2}\vec{g}\,t^2$ is the vertical displacement arising from the gravity. This deepens and gives a new insight to the meaning of the projectile problem as a superposition of two independent motions:

(i) constant velocity motion in the horizontal direction;
(ii) free-fall motion in the vertical direction with some initial velocity.

In addition, the time of flight $t$ is a parameter that established the necessary link between the two components, which are completely independent of each other. This aspect is important for the methodology presented in section 5.

At this point teachers can show that (2) and (3) can be obtained directly from the projection of the vectors presented in Fig. 2. However, the purpose of this section is to point out a method that analyses whole vector instead of its components. An important point of this methodology starts with the rule of addition of vectors as illustrated. This is relevant bearing in mind the difficulties of students of introductory physics with vectorial calculus. This vectorial analysis is a good complement to the traditional approach of the previous section and is easily generalized to problems with other initial conditions. So, let us address a second general example where $\vec{r}_0 \neq 0$.

*Example 2. A particle is thrown from the top of a building upward at an angle $\alpha$ to the horizontal and with an initial speed $v_0$. The height of the building is h. Determine the time of flight and the maximum distance from the building when the particle reaches the ground (Fig. 3).*

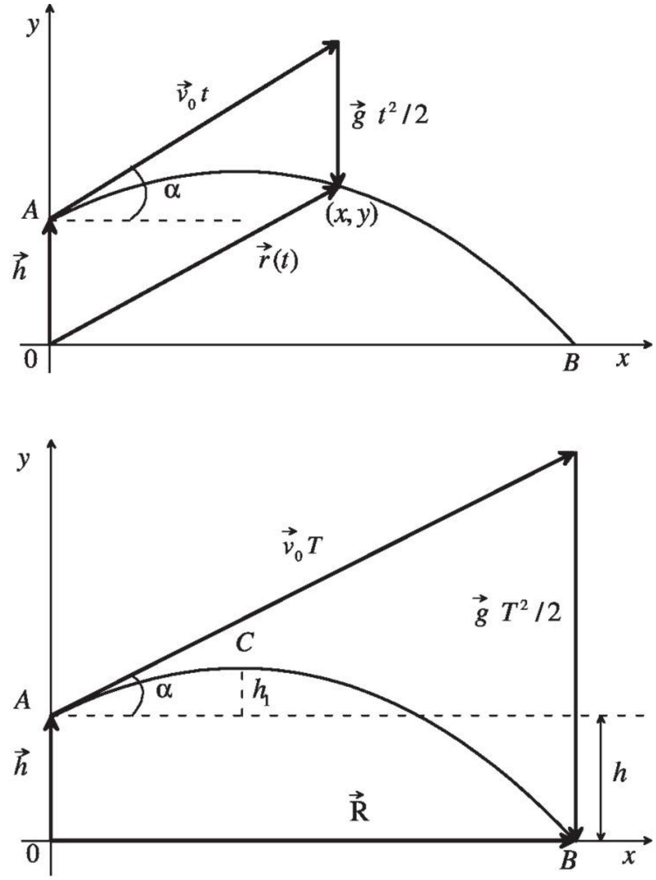


**FIGURE 3.** Kinematical quantities of a projectile problem at the instant *t* (upper) and at the total time of flight *T* (lower) where the range *R* is attained.

In this case the position vector at the instant $t$ is given by the addition of three vectors:

$$\vec{r}=\vec{h}+\vec{v}_0\,t+\frac{1}{2}\vec{g}\,t^2. \quad (9)$$

In Fig. 3, we illustrate this generalization showing the different displacements at a generic instant $t$ and at the time of flight $T$.

As the given quantities are: $v_0$, $\alpha$ and $h$, by using elementary trigonometry, the time of flight $T$ and the maximum range $R$ are easily calculated. To this purpose we write (see Fig. 3, lower part)

$$\frac{1}{2}g\,T^2-h=v_0\,T\sin\alpha, \quad (10)$$

$$R=v_0\,T\cos\alpha. \quad (11)$$

From these two equations we obtain the time of flight:

$$T=\frac{v_0 \sin\alpha}{g}\left(1+\left(1+2\,g\,h/v_0^2\sin^2\alpha\right)^{1/2}\right), \qquad (12)$$

and the maximum range:

$$R=\frac{v_0^2 \sin 2\alpha}{2g}\left(1+\left(1+2\,g\,h/v_0^2\sin^2\alpha\right)^{1/2}\right). \qquad (13)$$

In the limit $h=0$ the Eqs. (4) and (5) for time flight and the maximum range, written in the previous section, are recovered.

## IV. IMPULSE-MOMENTUM AND WORK-ENERGY THEOREMS: CONSERVATION OF MOMENTUM AND ENERGY

The impulse-momentum and work-energy theorems are tools that students usually appreciate to solve problems. However, teachers do not in general motivate students to look at these theorems to analyse projectile behavior. As a further step, we may introduce explicitly this point of view, showing that the conservation of energy can also be useful.

### A. Analytical development

The impulse of a force over time produces a change in the momentum $\vec{p}=m\vec{v}$, where *m* is the mass of the particle. It is defined by the integral of the applied force over the time interval 0 and *t*, allowing for the so-called impulse-momentum theorem:

$$\vec{I}=\int_0^t \vec{F}\,dt=\Delta\,\vec{p}, \qquad (14)$$

which is the integral form of Newton's second law.

Let us illustrate the application of this procedure to *example 2*. Substituting $\vec{F}=m\vec{g}$ in this theorem and simplifying, we find the standard kinematic expressions

$$v_x=v_{0x}=v_0\cos\alpha, \qquad (15)$$

$$g\,t+v_y=v_{0y}=v_0\sin\alpha. \qquad (16)$$

It may be noted that the first of these equations corresponds to the conservation of the momentum along the $x$ axis, which is a consequence of the absence of any force in the horizontal direction.

The work done by a force over displacement produces a change in the kinetic energy, *K*, of the particle upon it acts. The integral is now defined over a displacement, allowing for the work-energy theorem for the particle:

$$W=\int_0^{\vec{r}} \vec{F}.d\vec{r}=\Delta K. \qquad (17)$$

As the gravitational force is a conservative one, this is equivalent to the conservation of total mechanical energy of the system particle-Earth. In fact, a conservative force acts between the members of the system; the point of application of the force undergoes a displacement, and work is done by the force. The corresponding potential energy function $U(y)$ changes according to $W=-\Delta U$. So, we can write for the particular case of *example 2*, the law of conservation of energy

$$m\,g\,y+\frac{1}{2}m(v_x^2+v_y^2)=m\,g\,h+\frac{1}{2}m(v_{0x}^2+v_{0y}^2), \qquad (18)$$

where the zero point for potential energy is defined at $y=0$.

Since $v_x=v_{0x}$, the previous equation can be simplified:

$$g\,y+\frac{1}{2}v_y^2=g\,h+\frac{1}{2}v_{0y}^2. \qquad (19)$$

We remark that (18) expresses the conservation of total mechanical energy: $E(v_x,y,v_y)=E_1(v_x)+E_2(y,v_y)=$const. In fact, the total mechanical energy is given by $E(v_x,y,v_y)=\frac{1}{2}m(v_x^2+v_y^2)+mg\,y$ [1] coming from the summation of

$E_1(v_x)=\frac{1}{2}m v_x^2$ and

$E_2(y,v_y)=\frac{1}{2}m v_y^2+m\,g\,y$, both constants.

In this methodology we obtain the components of the velocity

$$v_x=v_0\cos\alpha, \qquad (20)$$

and

$$v_y=\pm\left(v_0^2\sin^2\alpha+2\,g\,(h-y)\right)^{1/2}, \qquad (21)$$

where $v_y$ is a function of *y*.

The last equations are very convenient to obtain the equation of the trajectory. In fact, since the velocity vector is tangential to the trajectory at every point, the differential equation of the trajectory of the particle follows directly from the requirement

$$\mathrm{d}\vec{r}\wedge\vec{v}=0, \qquad (22)$$

[1] In the Hamiltonian or Lagrangian formalism we say that *x* is a cyclic coordinate, which implies that the *x* component of the momentum is a constant of motion. The conservation of energy comes from the absence of *t* in the energy function.

involving the vector cross product of $\vec{v}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$, where $\hat{i}$ and $\hat{j}$ are the Cartesian unit vectors along $x$ and $y$, respectively.

This condition can be expressed as

$$v_y\,dx - v_x\,dy = 0. \tag{23}$$

In this case, the insertion of (20) and (21) into (23) allows to obtain

$$\left(v_0^2 \sin^2\alpha + 2g(h-y)\right)^{-1/2} dy - \frac{1}{v_0\cos\alpha}\,dx = 0, \tag{24}$$

which on integration yields

$$\left(v_0^2 \sin^2\alpha + 2g(h-y)\right)^{1/2} + \frac{g\,x}{v_0\cos\alpha} = \text{const.} \tag{25}$$

It is worth noting that if $h = 0$, then the equation of the parabolic trajectory (7) is recovered.

### B. Application

Now, let us consider a problem, adapted from Serway and Beichner [12], that shows as the combination of the vectorial analysis with the conservation of energy can be a very convenient tool.

*Problem 1. A particle of mass $m = 0.5\,kg$ is shot from point $A$ at the top of a building. The particle has an initial velocity $\vec{v}_0$ with a horizontal component of 30m/s. The height of the building is $h = 60\,m$ and the particle rises to a maximum height $h_1 = 20m$ at point C, and reaches the ground at point B (see Fig. 3). Determine, using the value $g = 9.8\,\text{m}\,\text{s}^{-2}$:*

- *(i) the vertical component of $\vec{v}_0$,*
- *(ii) the time of flight of the particle and the maximum distance from the building when the particle reaches the ground,*
- *(iii) the work done by the gravitational force on the particle during its motion from A to B, and*
- *(iv) the horizontal and vertical components of the velocity vector when the particle reaches B.*

*Solution*

(i) The conservation of energy (18) applied to the initial position $A$ and to point $C$ where $v_y = 0$, together with $v_x = v_{0x}$, allow to obtain the vertical component of the initial velocity:

$$v_{0y} = \sqrt{2\,g\,h_1} = 19.8\,\text{m/s}. \tag{26}$$

(ii) The time of flight and the maximum range can be obtained by geometry as illustrated in the lower part of Fig. 3 (see also (10) and (11)), giving $T$ = 6.1 s and $R$ = 181.8 m.

(iii) The work can be provided by the variation of potential energy, yielding

$$W_{AB} = -\left(U_B - U_A\right) = m\,g\,h = 294.0\,\text{J}. \tag{27}$$

(iv) Applying again the conservation of energy to points $A$ and $B$ it is straightforward to calculate the components of the velocity at position $B$:

$$v_y = -\sqrt{2\,g\,(h_1 + h)} = -39.6\,\text{m/s}, \tag{28}$$

and

$$v_x = 30\,\text{m/s}. \tag{29}$$

## V. APPROACH BASED ON FIRST INTEGRALS OF MOTION

For advanced students, we can go further giving a new insight into the equations obtained in the previous sections. As already referred, the projectile motion can be decomposed into two components characterized by four dynamical variables: $x$, $v_x$, $y$ and $v_y$. These dynamical variables, which specify the state of the system, are functions of time $t$, parameter which provides the necessary connection between the two components.

Meanwhile, there exist functions of the dynamical variables and, eventually of the time, the so-called integrals of motion [9], whose values remain constant during the motion, and depend only on the initial conditions. So, we present a methodology that allows to interpret the equations of the projectile problem as first integrals of motion, which corresponds to constants of motion. This method, here presented at an elementary level, was formulated in a more formal way [13], following a methodology proposed by Wittaker [14]. We remember that the traditional way to obtain constants of motion comes from a much more elaborated mathematical framework based on space-time symmetries and cyclic coordinates [9].

To discuss, from a pedagogical viewpoint, how constants of motion can be extracted and used to analyse the projectile behavior, let us consider the following illustrative problem [15], to be explicitly solved later.

*Problem 2. An elastic ball is dropped on a long inclined plane at point A. It bounces, hits the plane again, bounces, and so on. Let us label the distance between the points of the first and the second hit $d_{12}$, and the distance between the points of the second and the third hit $d_{23}$. Find the ratio $d_{12}/d_{23}$.*

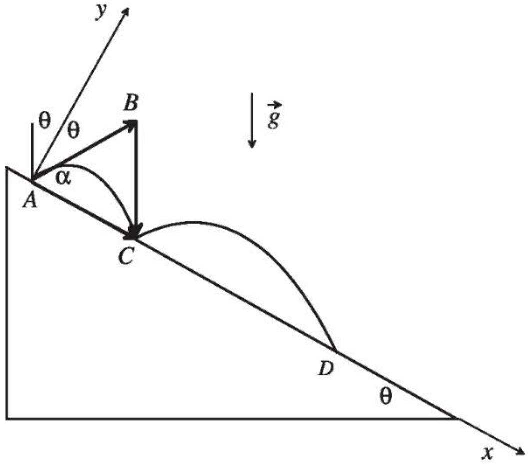


**FIGURE 4.** Trajectory of an elastic ball on an inclined plane. The triangle $ABC$ is isosceles. The magnitude of the vectors indicated are given by $AB=v_0 t_C$ and $BC=\frac{1}{2} g t_C^2$ , and $\theta$ and $\alpha$ are complementary angles.

### A. Analytical development

The referential indicated in Fig. 4 is the most convenient to solve the problem. According to this referential, the relevant information can be written as

$$\vec{v}_0 = v_0 \cos\alpha\,\hat{i} + v_0 \sin\alpha\,\hat{j}, \tag{30}$$

$$\vec{g} = g\cos\alpha\,\hat{i} - g\sin\alpha\,\hat{j}. \tag{31}$$

We start with the definitions:

$$\frac{dx}{dt}=v_x, \frac{dy}{dt}=v_y, \frac{dv_x}{dt}=g\cos\alpha, \frac{dv_y}{dt}=-g\sin\alpha. \tag{32}$$

As $t$ is a common parameter of (32), we can also write

$$\frac{dx}{v_x}=\frac{dy}{v_y}=\frac{dv_x}{g\cos\alpha}=\frac{dv_y}{-g\sin\alpha}=\frac{dt}{1}. \tag{33}$$

This leads to a set of rearrangements of the differential equations involved, which are going to be analyzed in detail. This procedure allows to find first integrals for this system which corresponds to constants of motion.

1. To obtain the first constant of motion $C_1$, we start with the relation

$$v_x\, dv_x = g\cos\alpha\, dx, \tag{34}$$

showing that, by direct integration,

$$C_1 = \frac{1}{2}v_x^2 - g\cos\alpha\, x = \frac{1}{2}v_0^2\cos^2\alpha. \tag{35}$$

2. Analogously, from equation

$$v_y\, dv_y = -g\sin\alpha\, dy, \tag{36}$$

we get, on integration, a second constant of motion

$$C_2 = \frac{1}{2}v_y^2 + g\sin\alpha\, y = \frac{1}{2}v_0^2\sin^2\alpha. \tag{37}$$

Before pursuing, it should be emphasised that the constants of motion $C_1$ and $C_2$ are actually compatible with the conservation of total mechanical energy: $E(x,v_x,y,v_y)=E_1(x,v_x)+E_2(y,v_y)=\text{const}$ , where $E_1 = mC_1 = \frac{1}{2}mv_x^2 - mg\cos\alpha\, x$ and $E_2 = mC_2 = \frac{1}{2}mv_y^2 + mg\sin\alpha\, y$, both constants.

3. Once again, from the relation

$$dv_x = g\cos\alpha\, dt, \tag{38}$$

we get a new constant of motion

$$C_3 = v_x - g\cos\alpha\, t = v_0\cos\alpha. \tag{39}$$

4. Analogously, from the equation

$$dv_y = -g\sin\alpha\, dt, \tag{40}$$

we obtain

$$C_4 = v_y + g\sin\alpha\, t = v_0\sin\alpha. \tag{41}$$

The equations of motion (39) and (41) correspond to the impulse-momentum theorem along the $x$ and $y$ directions, respectively. The interesting point is that these two constants of motion $C_3$ and $C_4$ depend explicitly on time $t$. This is intimately related to the presence of a force along both axes $x$ and $y$.

5. There are more three relations from (33):

$$v_x\, dv_y = -g\sin\alpha\, dx, \tag{42}$$

$$v_y\, dv_x = g\cos\alpha\, dy, \tag{43}$$

and

$$\frac{dv_x}{\cos\alpha} = -\frac{dv_y}{\sin\alpha}, \tag{44}$$

that yield the same conclusion:

$$C_5 = \frac{v_x}{\cos\alpha} + \frac{v_y}{\sin\alpha} = 2v_0. \tag{45}$$

Straightforward mathematical manipulations show that this equation can also be obtained by combining Eqs. (39) and (41).

6. Finally, the differential equation $v_x\,\mathrm{d}\,y - v_y\,\mathrm{d}\,x = 0$ is also contained in (33), which on integration gives the equation of the trajectory as already illustrated in the previous section.

The mathematical derivations presented here can be simplified in situations, like *Example 1*, where $v_x$ is itself a constant of motion [13].

So, starting with the dynamical equations, we can find constants of motion of this classical problem with two degrees of freedom. With some of these equations, it is easy to solve the problem proposed. The remaining equations can be used to check the solution of the problem we are dealing with.

**B. Solution of problem 2**

Applying (41) to the flight phase between first and second hit one can show that the time to get the maximum distance from the inclined plane is given by $v_0 / g$. So, the time spent in each flight is $t_C = 2v_0 / g$. The vectorial construction in Fig. 4, which shows that the triangle $ABC$ is isosceles, confirms this result.

At point *C*, the components of the velocity can be calculated through (39) and (41), yielding

$$v_x = 3v_0 \cos\alpha, \tag{46}$$

and

$$v_y = -v_0 \sin\alpha. \tag{47}$$

After the elastic ball-plane collision the takeoff velocity components are $v_x = 3v_0 \cos\alpha$ and $v_y = v_0 \sin\alpha$. This indicates that in the next flight phase the maximum distance from the inclined plane is the same of the first flight and the range is larger.

The displacement along the plane follows from (35) and (46):

$$d_{12} = \frac{1}{2}\frac{v_x^2}{g\cos\alpha} - \frac{1}{2}\frac{v_0^2}{g}\cos\alpha = \frac{4v_0^2}{g}\cos\alpha. \tag{48}$$

Equation (37) confirms that *y*=0 at point C.

At point *D*, it is easy to show that the components of the velocity are:

$$v_x = 5v_0 \cos\alpha, \tag{49}$$

and

$$v_y = -v_0 \sin\alpha. \tag{50}$$

This result indicated that the time of each flight is, in fact, the same.

Using a procedure similar to the previous one we obtain

$$d_{23} = \frac{8v_0^2}{g}\cos\alpha, \tag{51}$$

showing that

$$\frac{d_{12}}{d_{23}} = \frac{1}{2}. \tag{52}$$

The extra Eq. (45) can be used to check the values obtained for the components of the velocity at points *C* and *D*. We invite the reader to obtain other interesting aspects of this challenge problem as, for instance, the location of the maximum distance from the plane attained in each flight.

## VI. CONCLUSIONS

We have discussed the projectile motion, where the usual assumption of constant acceleration of gravity and absence of air resistance have been considered. We started with two typical examples which are used as exploratory tools. We have verified that the vector method based on the whole position vector gives significant pedagogical advantage when compared with the decomposition method alone. If we look at several textbook problems, we verify that some aspects can be easily solved by the conjugation of this and other procedures.

For advanced students, the methodology based on the notion of first integrals of motion, gives an overview on this topic, showing the interplay and complementarity of the different descriptions. It can also provide the starting point for the discussion of constants of motion of any classical system. As a matter of fact, these approaches in a familiar problem help students to understand the dynamical content of the conservation laws, and can establish a natural bridge to introduce this topic of fundamental importance in several areas of physics.

In conclusion, the nontraditional approaches invite students to elaborate a general procedure and, after that, they can easily solve the problem. In this way they do not worry about "memorized equations". We verified that this procedure enhances the physical content of the subject when compared with the traditional method. In addition, it is worthwhile showing that this approach can be carried out even with a simple high school level of algebra, which most students of intermediate physics course are supposed to be familiar with. In fact, it should be emphasized that in the first examples no mathematics beyond elementary vectorial calculus and trigonometry was employed. Meanwhile, the more advanced method allows to focus on supplying the mathematical framework of the linear first-order differential equations. The advantage is that this is done in a context that the students were already familiar with from a physical point of view.

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