Dragging a string over a step

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Abstract

A simple problem in Newtonian mechanics is considered. The problem consists in finding the maximum value of the length x_{UP} of the portion of string slowly dragged on a step of height h, when the string itself is initially placed to match the vertical profile of the step, the remaining part lying on the ground and the final portion being in static equilibrium during the dragging process. A straightforward analysis is required to find the solution. The problem can be proposed in a lecture or a demonstration in class on the role played by the coefficient of static friction in mechanics.

Keywords: Classical Mechanics teaching, static coefficient of friction.

Resumen

Un simple problema de la mecánica newtoniana es considero. El problema consiste en encontrar el valor máximo de la longitud x_{UP} de la porción de cadena lentamente arrastrado en un paso de altura h, cuando la propia cadena se coloca inicialmente para que coincida con el perfil vertical del paso, la parte restante en el suelo y la porción final de estar en equilibrio estático durante el proceso de arrastre. Un análisis simple se requiere para encontrar la solución. El problema puede ser propuesto en una conferencia o una demostración en clase sobre el papel desempeñado por el coeficiente de fricción estática en la mecánica.

Palabras clave: Enseñanza de la Mecánica clásica, coeficiente de fricción estática.

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I. THE PROBLEM

Consider a rounded step of height h. A string of length L and linear mass density λ is initially placed in such a way that one end follows the vertical profile of the step and the remaining part lies on an horizontal rough surface as shown in Fig. 1a.

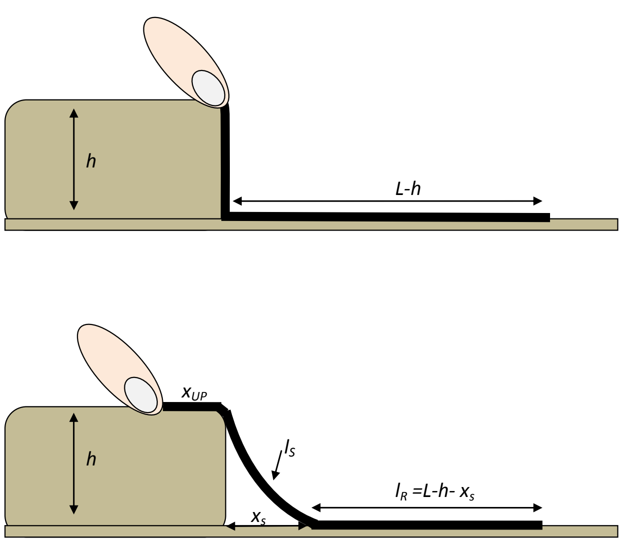


FIGURE 1. a) A portion of a string of length L lies over the vertical profile of a rounded step, the remaining part lies on a horizontal rough surface. The end of the vertical portion is held by a finger at the edge of the step. b) The string is slowly dragged by the finger over the step, while the other end is not moving: the suspended part of the string is seen to have length l_s.

The coefficient of static friction between the surface and the string is μ_S [1]. The string is then slowly dragged from the upper end over the step with a finger, as shown in Fig. 1b, until the other end starts moving.

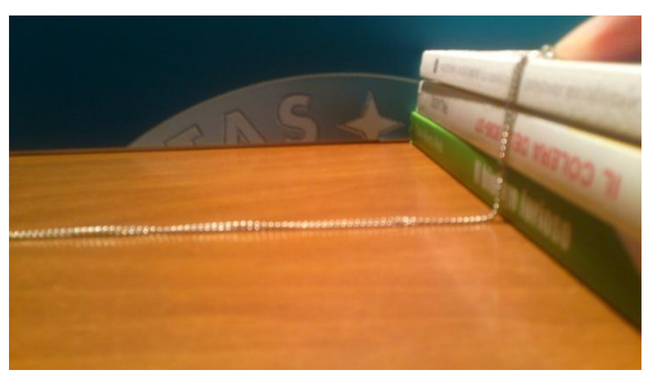


FIGURE 2. a) A small necklace of length L lies over the vertical profile of a pile of books, the remaining part lies on a horizontal surface. The end of the vertical portion is held by a finger at the edge of the step. b) The string is slowly dragged by the finger over the pile of books, while the other end is not moving: the suspended part of the string is seen to take the form of a catenary.

We notice that the portion of the suspended part of the string takes a form of a catenary. Just before the portion of the string lying on the horizontal surface starts moving, we record the value of the horizontal distance x_{UP} (see Fig. 1b) and the distance x_S between the closest point of contact of the string to the lower edge of the step. The evidence that the suspended portion of the string, of length l_S, takes the form of a catenary is well reproduced in Figs. 2a-b, where a small necklace is dragged above a pile of books. Notice that by varying the number of books in the pile one can change the value of h, so that the ratio L/h can be varied by keeping either h or L fixed.

II. THE SOLUTION

By considering the schematic diagrams in Figs. 3a and 3b, describing the forces acting on the suspended and horizontal portion of the string, respectively, we may find the conditions for static equilibrium. In particular, for the suspended portion of the string, by setting the resultant force equal to zero [1], we have:

$$\begin{aligned} T_A \cos\theta &= T_B \\ T_A \sin\theta &= m_S g \end{aligned} , \qquad (1)$$

Where $m_S=\lambda l_S$, and T_A and T_B are the moduli of the tensions at the cuts shown in Fig. 3a.

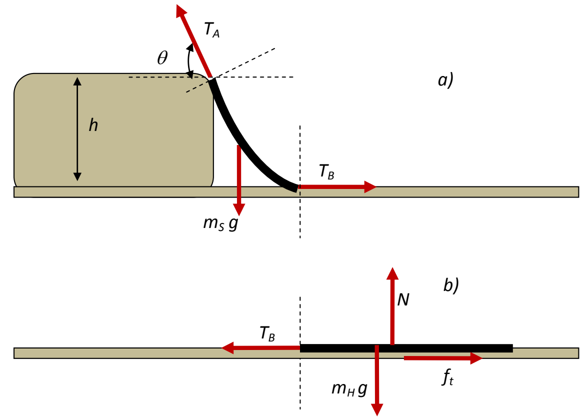


FIGURE 3. a) Suspended portion of the string. Only tensions at both orthogonal cuts are considered. b) Horizontal portion of the string: the friction f_t is sufficient to maintain the system in equilibrium.

On the other hand, for the horizontal portion we write:

$$\begin{aligned} f_t &= T_B \\ N &= m_H g \end{aligned} , \qquad (2)$$

where $m_H=\lambda l_R=\lambda(L-h-x_S)$ is the mass of the portion of the string lying on the horizontal surface, f_t and N are the moduli of the friction force and of the normal reaction, respectively. By now introducing the phenomenological relation $f_t \le \mu_S N$ valid for static equilibrium of the system, we consider the case of incipient motion. Therefore, by eliminating the tensions T_A and T_B by means of (1) and (2) and by setting $f_t = \mu_S N$, we obtain:

$$\tan\theta = \frac{l_S}{\mu_S l_R}. \qquad (3)$$

The expression for l_S can be obtained by the equation of the catenary for the suspended portion of the curve. In fact, by fixing the origin of the x-axis at the same point B where the orthogonal cut to obtain tension T_B is made (see Figs. 3a-b), by taking x positive toward the left, the catenary equation can be written as follows [2]:

$$y(x) = \mu_S l_R [\cosh\left(\frac{x}{\mu_S l_R}\right) - 1]. \qquad (4)$$

Therefore, since $dl = (1+y'^2)^{1/2}\,dx$, y' being the derivative of y with respect to x, l_S can be obtained by the following integration:

$$l_S = \int_0^{x_S} \sqrt{1+y'^2}\,dx = \mu_S l_R \sinh\left(\frac{x_S}{\mu_S l_R}\right). \qquad (5)$$

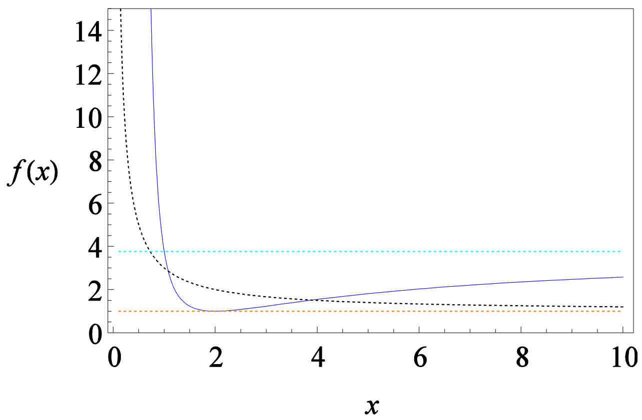


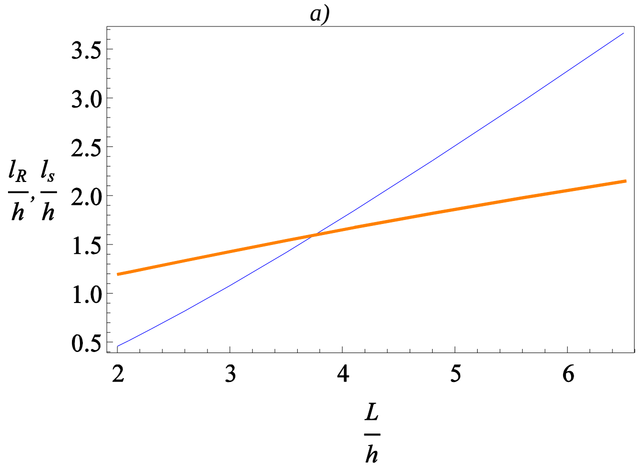
FIGURE 4. Graphical representation of the curves $f_1(x)$ and $f_2(x)$ corresponding, respectively, to the left hand side (blue full line) and to the right hand side (dotted black line) of (8) for l=3.0 and μ_S=0.5. The cyan dotted line represents the right horizontal asymptote of $f_1(x)$, while the orange dotted line $f(x)=1$ represents the right horizontal asymptote of $f_2(x)$. Notice that $f(x)=1$ is tangent to $f_1(x)$ at its minimum point at $x=l-1$. Finally notice that only the left intersection at x_1 between the curves $f_1(x)$ and $f_2(x)$ gives a meaningful solution for (8), being $x_1<l$.

Eq. (4) can be used to obtain a relation between x_S and h, by setting $y(x_S)=h$, so that:

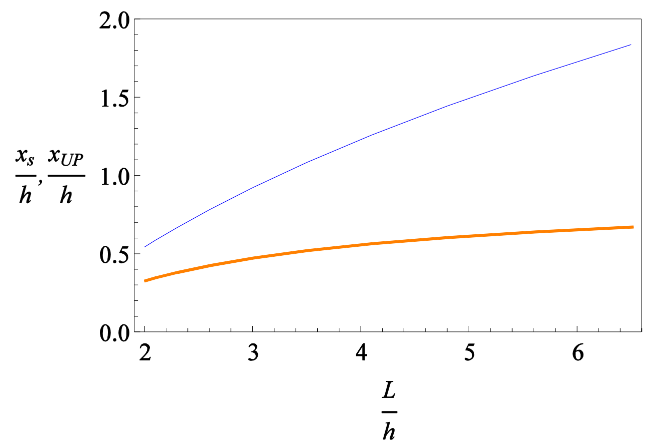
$$h=\mu_S l_R[\cosh\left(\frac{x_S}{\mu_S l_R}\right)-1]\cdot \qquad (6)$$

Recalling now that $\cosh^2 x-\sinh^2 x=1$, by combining (3), (5), and (6), and by setting $l_R=L-h-x_S$, we have:

a)



b)



c)

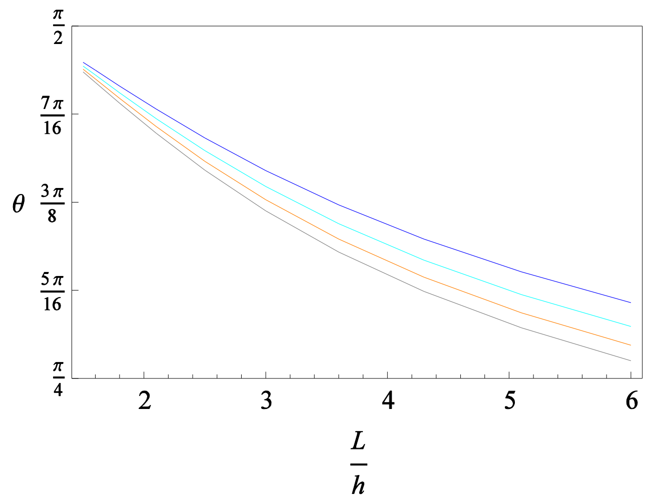


FIGURE 5. Graphical representation of the dependence of the quantities l_R, l_S, x_s, x_{UP}, and θ as functions of $l=L/h$. In particular, in *a)* l_R (blue line), and l_S (orange line) vs. l curves are reported for μ_S=0.5. In *b)* x_s (blue line), and x_{UP} (orange line) vs. l curves are shown for μ_S=0.5. Finally, in *c)* θ vs. l curves are plotted for μ_S=0.4 (blue line), μ_S=0.5 (cyan line), μ_S=0.6 (orange line), and μ_S=0.7 (gray line).

$$\cosh\left(\frac{L-h}{\mu_S l_R}-\frac{1}{\mu_S}\right)=1+\frac{h}{\mu_S l_R}; \qquad (7a)$$

$$\cos\theta=\left(1+\frac{h}{\mu_S l_R}\right)^{-1}. \qquad (7b)$$

By solving numerically (7a) for l_R, we can obtain θ from (7b) and, by the knowledge of the latter two quantities, we can get l_S, $x_{UP}=L-l_S-l_R$, and x_S.

III. NUMERICAL RESULTS

We can solve Eq. (7a) numerically for $x=l_R/h$ in terms of the parameters μ_S and $l=L/h$. Let us thus write Eq. (7a) as follows:

$$\cosh\left(\frac{l-1}{\mu_S x}-\frac{1}{\mu_S}\right)=1+\frac{1}{\mu_S x}. \qquad (8)$$

The functions $f_1(x)$ and $f_2(x)$ on the right and left hand side of Eq. (8), respectively, are represented in Fig. 4 for l=3.0 and μ_S=0.5. By the rule we can argue from Fig. 4, for which only the left intersection represents the meaningful solution to (8), we obtain the solutions in terms of l, reported in Figs. 5a-c for fixed values of μ_S. Notice that, for increasing values of the normalized length of the string $l=L/h$, the quantities l_R, l_S, x_S, x_{UP} increase. However, as shown in Fig. 5a, the derivative of l_R with respect to l, for a given value of the latter normalized quantity, is always greater than the derivative of l_S for the same value of l. Similarly, in Fig. 5b we may notice that the derivative of x_S with respect to l, for a given value of the latter normalized quantity, is always greater than the derivative of x_{UP} for the same value of l. In Fig. 5c, finally, we may notice that the derivatives of all θ vs. l curves are negative for any value of l in the represented range of values of the latter quantity.

The behavior of the curves shown in Figs. 5a-b can be justified by the higher value the friction force obtained by increasing l, μ_S being kept constant.

In Figs. 6a-c we show the quantities l_R, l_S, x_S, x_{UP}, and θ in terms of the coefficient of static friction μ_S for fixed values of l. As it can be noted from the l-dependence of the distances l_S, x_S, and x_{UP}, a positive derivative with respect to μ_S is detectable in Figs. 6a-b, differently from the decreasing behavior of l_R for increasing values of μ_S in Fig. 6a. In Fig. 6c one notices that all curves attain a negative derivative. Furthermore, in the same Fig. 6c one may see that, for a fixed value of l, the angles θ are lower as μ_S increases from 0.4 to 0.7, coherently with what shown in Fig. 5c.

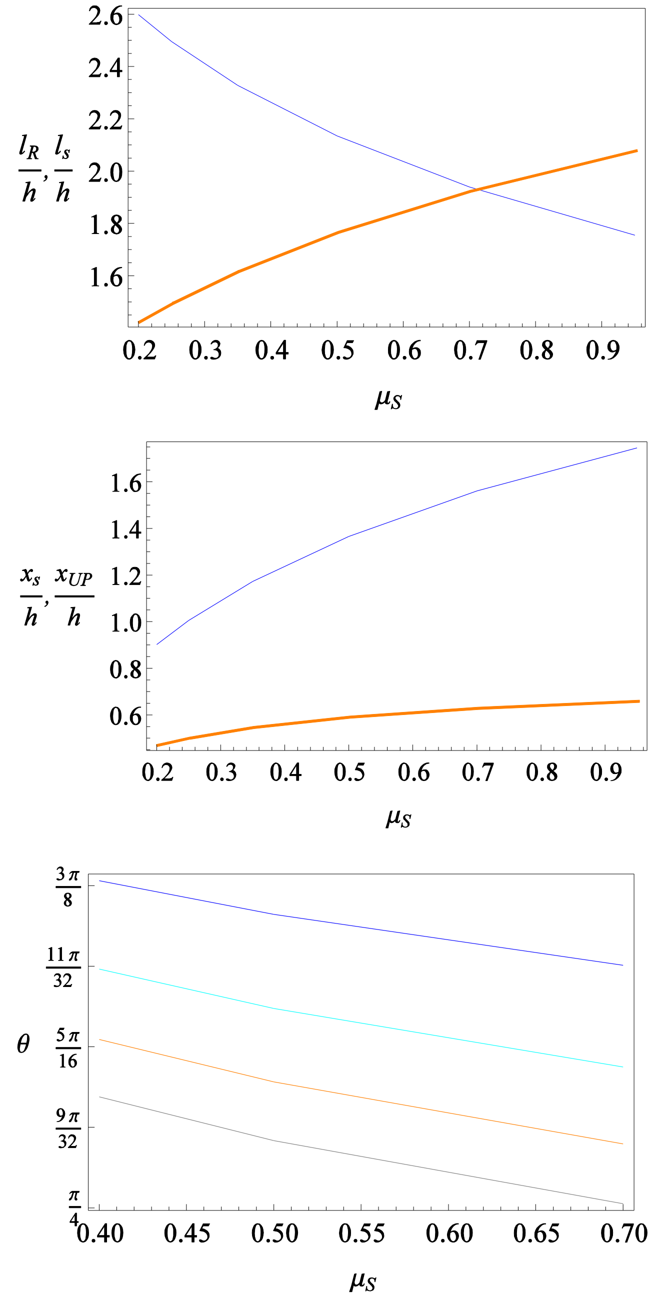


FIGURE 6. Graphical representation of the dependence of the quantities l_R, l_S, x_s, x_{UP}, and θ as functions of μ_S. In particular, in *a)* l_R (blue line), and l_S (orange line) vs. μ_S curves are reported for l=4.5. In *b)* x_s (blue line), and x_{UP} (orange line) vs. μ_S curves are shown for l=4.5. Finally, in *c)* θ vs. μ_S curves are plotted for l=3.5 (blue line), l=4.5 (cyan line), l=5.5 (orange line), and l=6.5 (gray line).

IV. CONCLUSIONS

By studying a rather straightforward problem, we are able to illustrate the role played by the coefficient of static friction in Newtonian mechanics. The solution to the problem can be found by elementary principles in mechanics and results can be represented graphically by means of numerical analysis. Furthermore, given the rather simple experimental setup required to reproduce the system in real terms, a classroom demonstration experiment can be performed to illustrate the meaning of the coefficient of static friction in mechanics. The content of the present work can be part of a lecture addressed to advanced high-school students or to first-year college students.

REFERENCES

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[2] Beer, F. P., Johnston, E. R., *Vector Mechanics for Engineers*, 3rd Ed. (McGraw-Hill, New York, 1977), pp. 297-299.