Dragging a string over a step

Roberto De Luca  
Dipartimento di Fisica “E. R. Caianiello”, Università degli Studi di Salerno.  
Via Ponte don Melillo, 84084 Fisciano (SA), Italy.  
E-mail: rdeluca@unisa.it

(Received 6 February 2012, accepted 29 February 2012)

Abstract

A simple problem in Newtonian mechanics is considered. The problem consists in finding the maximum value of the length $x_{UP}$ of the portion of string slowly dragged on a step of height $h$, when the string itself is initially placed to match the vertical profile of the step, the remaining part lying on the ground and the final portion being in static equilibrium during the dragging process. A straightforward analysis is required to find the solution. The problem can be proposed in a lecture or a demonstration in class on the role played by the coefficient of static friction in mechanics.

Keywords: Classical Mechanics teaching, static coefficient of friction.

I. THE PROBLEM

Consider a rounded step of height $h$. A string of length $L$ and linear mass density $\lambda$ is initially placed in such a way that one end follows the vertical profile of the step and the remaining part lies on an horizontal rough surface as shown in Fig. 1a.

![Figure 1a](image1.png)

The coefficient of static friction between the surface and the string is $\mu_s$ [1]. The string is then slowly dragged from the upper end over the step with a finger, as shown in Fig. 1b, until the other end starts moving.

![Figure 1b](image2.png)
We notice that the portion of the suspended part of the string takes a form of a catenary. Just before the portion of the string lying on the horizontal surface starts moving, we record the value of the horizontal distance \( x_{u/p} \) (see Fig. 1b) and the distance \( x_s \) between the closest point of contact of the string to the lower edge of the step. The evidence that the suspended portion of the string, of length \( l_s \), takes the form of a catenary is well reproduced in Figs. 2a-b, where a small necklace is dragged above a pile of books. Notice that by varying the number of books in the pile one can change the value of \( h \), so that the ratio \( L/h \) can be varied by keeping either \( h \) or \( L \) fixed.

II. THE SOLUTION

By considering the schematic diagrams in Figs. 3a and 3b, describing the forces acting on the suspended and horizontal portion of the string, respectively, we may find the conditions for static equilibrium. In particular, for the suspended portion of the string, by setting the resultant force equal to zero [1], we have:

\[
T_A \cos \theta = T_B,
\]

\[
T_A \sin \theta = m_s g,
\]

(1)

Where \( m_s = \lambda l_s \), and \( T_A \) and \( T_B \) are the moduli of the tensions at the cuts shown in Fig. 3a.

On the other hand, for the horizontal portion we write:

\[
f_r = T_A,
\]

\[
N = m_H g,
\]

(2)

where \( m_H = \lambda l_H = \lambda (L-h-x_s) \) is the mass of the portion of the string lying on the horizontal surface, \( f_r \) and \( N \) are the moduli of the friction force and of the normal reaction, respectively. By now introducing the phenomenological relation \( f_r \leq \mu_s N \) valid for static equilibrium of the system, we consider the case of incipient motion. Therefore, by eliminating the tensions \( T_A \) and \( T_B \) by means of (1) and (2) and by setting \( f_r = \mu_s N \), we obtain:

\[
\tan \theta = \frac{l_s}{\mu_s l_H}.
\]

(3)

The expression for \( l_s \) can be obtained by the equation of the catenary for the suspended portion of the curve. In fact, by fixing the origin of the \( x \)-axis at the same point \( B \) where the orthogonal cut to obtain tension \( T_B \) is made (see Figs. 3a-b), by taking \( x \) positive toward the left, the catenary equation can be written as follows [2]:

\[
y(x) = \mu_s l_H [\cosh\left(\frac{x}{\mu_s l_H}\right) - 1].
\]

(4)

Therefore, since \( dl = (1+y'^2)^{1/2} \, dx \), \( y' \) being the derivative of \( y \) with respect to \( x \), \( l_s \) can be obtained by the following integration:

\[
l_s = \frac{1}{\mu_s l_H} \int_0^x \sqrt{1+y'^2} \, dx = \mu_s l_H \sinh\left(\frac{x}{\mu_s l_H}\right).
\]

(5)
Eq. (4) can be used to obtain a relation between \( x_S \) and \( h \), by setting \( y(x_S) = h \), so that:

\[
h = \mu_s l_R \left[ \cosh \left( \frac{x_S}{\mu_s l_R} \right) - 1 \right].
\]

Recalling now that \( \cosh^2 x - \sinh^2 x = 1 \), by combining (3), (5), and (6), and by setting \( l_R = L - h - x_S \), we have:

\[
\cosh \left( \frac{L - h}{\mu_s l_R} \right) - \frac{1}{\mu_s l_R} = 1 + \frac{h}{\mu_s l_R}.
\]

By solving numerically (7a) for \( l_R \), we can obtain \( \theta \) from (7b) and, by the knowledge of the latter two quantities, we can get \( l_S, x_{UP} = L - l_S - l_R, \) and \( x_S \).
Roberto De Luca

III. NUMERICAL RESULTS

We can solve Eq. (7a) numerically for $x=l_\theta/h$ in terms of the parameters $\mu_\theta$ and $l=L/h$. Let us thus write Eq. (7a) as follows:

$$\cosh\left(\frac{l-1}{\mu_\theta x} - \frac{1}{\mu_\theta}\right) = 1 + \frac{1}{\mu_\theta x}.$$  \(8\)

The functions $f_1(x)$ and $f_2(x)$ on the right and left hand side of Eq. (8), respectively, are represented in Fig. 4 for $l=3.0$ and $\mu_\theta=0.5$. By the rule we can argue from Fig. 4, for which only the left intersection represents the meaningful solution to (8), we obtain the solutions in terms of $l$, reported in Figs. 5a-c for fixed values of $\mu_\theta$. Notice that, for increasing values of the normalized length of the string $l=L/h$, the quantities $l_K$, $l_S$, $x_S$, $x_{UP}$ increase. However, as shown in Fig. 5a, the derivative of $l_K$ with respect to $l$, for a given value of the latter normalized quantity, is always greater than the derivative of $l_S$ for the same value of $l$. Similarly, in Fig. 5b we may notice that the derivative of $x_S$ with respect to $l$, for a given value of the latter normalized quantity, is always greater than the derivative of $x_{UP}$ for the same value of $l$. In Fig. 5c, finally, we may notice that the derivatives of all $\theta$ vs. $l$ curves are negative for any value of $l$ in the represented range of values of the latter quantity.

The behavior of the curves shown in Figs. 5a-b can be justified by the higher value the friction force obtained by increasing $l$, $\mu_\theta$ being kept constant.

In Figs. 6a-c we show the quantities $l_K$, $l_S$, $x_S$, $x_{UP}$, and $\theta$ in terms of the coefficient of static friction $\mu_\theta$ for fixed values of $l$. As it can be noted from the l-dependence of the distances $l_S$, $x_S$, and $x_{UP}$, a positive derivative with respect to $\mu_\theta$ is detectable in Figs. 6a-b, differently from the decreasing behavior of $l_K$ for increasing values of $\mu_\theta$ in Fig. 6a. In Fig. 6c one notices that all curves attain a negative derivative. Furthermore, in the same Fig. 6c one may see that, for a fixed value of $l$, the angles $\theta$ are lower as $\mu_\theta$ increases from 0.4 to 0.7, coherently with what shown in Fig. 5c.

IV. CONCLUSIONS

By studying a rather straightforward problem, we are able to illustrate the role played by the coefficient of static friction in Newtonian mechanics. The solution to the problem can be found by elementary principles in mechanics and results can be represented graphically by means of numerical analysis. Furthermore, given the rather simple experimental setup required to reproduce the system in real terms, a classroom demonstration experiment can be performed to illustrate the meaning of the coefficient of static friction in mechanics. The content of the present work can be part of a lecture addressed to advanced high-school students or to first-year college students.

REFERENCES
