Obliquely aligned bouncing ball experiment

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Abstract
We discuss the two-ball bounce problem of obliquely aligned balls. If one drops two balls without a rod which keeps two balls vertical, the upper ball obtains the horizontal component of its velocity after the sequential bouncing. Finally, it lands on the ground far from the bouncing point. As a practice for students in calculus-based Physics course, we show the flying distance of the upper ball and find the optimal initial angle of two balls. A tennis ball on a handball of size II give an example of the optimal angle to reach farthest.

Keywords: Classical Mechanics, conservation of momentum, Physics Education.

Resumen
Discutimos el problema del rebote de dos bolas de bolas alineadas oblicuamente. Si se dejan caer dos bolas, sin una barra que mantenga a las dos bolas verticales, la bola superior obtiene el componente horizontal de su velocidad, después del rebote secuencial. Finalmente, aterriza en el suelo lejos del punto de rebote. Como práctica para los estudiantes del curso de Física basado en el cálculo, mostramos la distancia de vuelo de la bola superior, y encontramos el ángulo inicial óptimo de las dos bolas. Una pelota de tenis, balonmano, de tamaño II, da un ejemplo del ángulo óptimo para llegar más lejos.

Palabras clave: Mecánica Clásica, conservación del momento, Enseñanza de la Física.

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I. INTRODUCTION

The velocity amplification problem of two vertically bouncing balls has been studied by many authors so far [1, 2, 3, 4, 5, 6, 7]. If two vertically aligned superballs are dropped simultaneously, the upper ball can bounce above the starting height, after sequential bouncing of balls and floor.

One finds that the maximum velocity gain of the upper ball before and after the collision, is three by making use of the momentum conservation law with elastic collisions [2, 3]. On the other hand, non-linear spring model [3] and energy-loss spring model [6] were also considered to explain the experimental results.

When performing the experiment in classroom, to show students this phenomena with two balls, $e.\ g.$ tennis ball on basketball, we sometimes make mistakes, because two balls are not always aligned in exactly vertical position. As a result, the upper tennis ball obtains horizontal component of its velocity, and lands somewhere far from teacher.

In this paper, we discuss the flying distance in such a case, and find the condition to obtain the largest flying distance. In order to simplify calculations below, we assume that, every chain collision can be described by Independent Collision Model (ICM) [3, 7] with elastic collisions.

We find that a tennis ball, on a handball of size II can be an example to realize the optimal situation, if mass ratio of these balls is nearly zero. As mathematical techniques rotation of axis, and the extremum condition including trigonometric functions, are required in this calculation.

Therefore, it will be helpful for students in calculus-based physics class.

II. OBLIQUELY ALIGNED BOUNCING BALL EXPERIMENT

The Figure 1 shows the setup of the problem. Ball A (mass m) and Ball B (mass M), inclined at an angle θ (assuming $0 \leq \theta \leq \pi/2$), are freely falling due to gravity. Ball B bounces on the ground, and Ball A and B, make an elastic collision with each other at relative velocity $2v_0$.

After the collision, Ball A and B fly into the air, with the velocities $\vec{v} = (v_x, v_y)$ and $\vec{V} = (V_x, V_y)$, respectively.

The motion of Ball A is described by equations of projectile motion, of the initial velocity $\vec{v}$ and angle $\tan\phi = v_x/v_y$, and it reaches the distance $L = 2v_xv_y/g$. The problem is to find the conditions to maximize L.

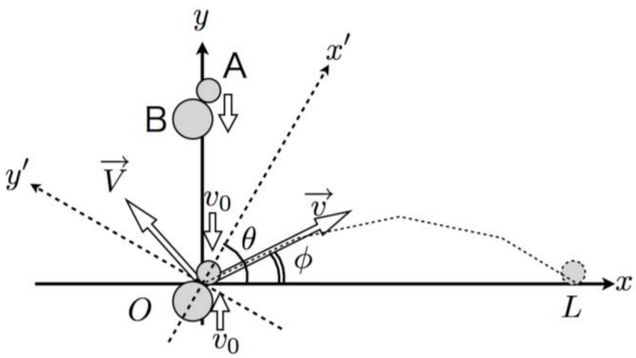

FIGURE 1. Schematic diagram of the obliquely aligned bouncing ball experiment.

Before and after the collision of two balls, the momentum conservation law in two dimensions are satisfied.

$$MV_x + mv_x = 0, \quad (1)$$

$$MV_y + mv_y = Mv_0 - mv_0, \quad (2)$$

Next, in order to discuss the coefficient of restitution, we make a transformation of coordinate from (x, y) to (x', y') described by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (3)$$

Therefore, the velocity vectors $\vec{v}$ and $\vec{V}$ in the (x', y') coordinate system are transformed by Equation (3), from those of the (x, y)-coordinate system, such as:

$$\begin{pmatrix} v'_x \\ v'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}. \quad (4)$$

Since we are assuming the collisions to be elastic, the coefficient of restitution between the balls along the x'direction is unity:

$$1 = -\frac{(V_x - v_x)\cos\theta + (V_y - v_y)\sin\theta}{2v_0 \sin\theta}, \quad (5)$$

where the numerator is the relative velocity in the (x', y') system: $V'_x - v'_x$.

Finally, since the relative velocity of the y' direction is invariant under the collision, should be satisfied.

$$2v_0 \cos\theta = -(V_x - v_x)\sin\theta + (V_y - v_y)\cos\theta \quad (6)$$

Solving Equations (1), (2), (5), (6), we obtain:

$$v_x = \frac{4}{1+\mu} v_0 \sin\theta\cos\theta, \quad (7)$$

$$v_y = \frac{4}{1+\mu} v_0 \sin^2\theta - v_0, \quad (8)$$

$$V_x = -\frac{4\mu}{1+\mu} v_0 \sin\theta\cos\theta, \quad (9)$$

$$V_y = -\frac{4\mu}{1+\mu} v_0 \sin^2\theta + v_0, \quad (10)$$

where we have introduced a new parameter $\mu = m/M \geq 0$.

As a consistency check, we find that $\theta = \pi/2$, gives the well-known results for ordinary stacked ball experiments with $v_x = V_x = 0$.

$$v_y = \frac{3-\mu}{1+\mu} v_0, \quad (11)$$

$$V_y = \frac{1-3\mu}{1+\mu} v_0, \quad (12)$$

From Equation (11), Ball A obtains the maximal speed $3v_0$ for $M \gg m$ $(\mu \to 0)$.

Now, we discuss the condition to maximize L.

First $v_y \geq 0$ should be required for Ball A to fly into the air after the collision. From Equation (8), it gives;

$$\theta \geq \frac{1}{2}\cos^{-1}\left(\frac{1-\mu}{2}\right). \quad (13)$$

The flying distance of Ball A, L, is easily calculated by Equations (7) and (8). We introduce a dimensionless parameter defined as $\tilde{L} = L/(v_0^2/g)$, given by:

$$\tilde{L} = \frac{4\sin 2\theta}{(1+\mu)^2}\,(1 - \mu - 2\cos 2\theta), \quad (14)$$

Where, v_0^2/g is the flying distance for the initial velocity v_0, with launching angle $\phi = 45°$.

The Figure 2 shows the allowed region for μ and θ, derived from Equation (13), which implies:

$$0 \leq \mu \leq 3, \quad 30° \leq \theta \leq 90°. \quad (15)$$

Notice that, two regions of Equation (15) are not independently allowed. From contour plot of $\tilde{L}(\theta, \mu)$ in Figure 2, one can see that, as a general behavior, smaller μ and $\theta \simeq 65°$ gives larger $\tilde{L}$. From Equation (14), one can verify that $\tilde{L}$ is monotonically decreasing function of μ, by calculating $d\tilde{L}/d\mu$; therefore the condition $\mu = 0$ has to be satisfied for getting the largest $\tilde{L}$. For $\mu = 0$, the extremum condition:

$$\frac{d\tilde{L}(\mu = 0)}{d\theta} = 8(\cos 2\theta - 2\cos 4\theta) = 0, \quad (16)$$

gives the maximal angle $\theta_{\max}$, given by:

$$\theta_{\max} = \frac{1}{2}\cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right) = 63.2°. \quad (17)$$

In this case, from Equations (7) and (8), Ball A obtains the velocity:

$$v_x = \frac{1}{2}\sqrt{\frac{15+\sqrt{33}}{2}}v_0, \quad v_y = \frac{3+\sqrt{33}}{4}v_0, \quad (18)$$

Which corresponds to:

$$|\vec{v}| = \sqrt{\frac{9+\sqrt{33}}{2}}\, v_0 = 2.72 v_0, \quad (19)$$

and

$$\phi_{\max} = \tan^{-1}\left(\frac{3+\sqrt{33}}{\sqrt{2(15+\sqrt{33})}}\right) = 53.6°. \quad (20)$$

Finally, we obtain the maximal flying distance, by making use of Eq. (18), as:

$$\tilde{L}_{\max} = \frac{v_x v_y}{v_0^2} = 7.04. \quad (21)$$

Since the maximal angle $\theta_{\max}$ is given by Equation (17) for $\mu = 0$, now we discuss how to realize such a situation. Let us consider free fall of two balls: Ball A with radius r and Ball B with radius R. If one put these balls on the wall, as shown in Figure 3, its angle θ above the horizontal satisfies:

$$\cos\theta = \frac{R-r}{R+r}. \quad (22$$

If Equation (22) satisfies the maximal condition, Equation (17), we obtain:

$$\frac{R}{r} = 2.64, \quad (23)$$

Which corresponds to tennis ball ($r = 3.27 - 3.43$cm) [8], and handball of size II ($R = 8.59 - 8.91$cm) [9].

Since handball is not infinitely heavy, and tennis ball is not massless ($\mu = 0.149 - 0.183$ in real case), the calculation above must be modified in the real experiments. However, this calculation performed here, will be a good exercise for students to understand two-dimensional collision phenomena.

III. CONCLUSIONS

We have discussed here the two-ball bounce problem with obliquely aligned balls, as a practice for students in calculus-based Physics course, the flying distance and its optimal initial angle of two balls, can be calculated by simple linear algebra, and solving extremum condition of trigonometric functions. One can obtain the optimal initial angle to be 63.2°. Next problem is to find how to realize the angle. By simple geometrical calculation, we found that a tennis ball on a handball of size II give such a situation. This will be a good exercise to understand two-dimensional collisions.

There are more Physical models of solving the two-ball bounce problem, than by assuming Independent Collision Model (ICM) used in this paper. Theoretical and experimental analysis of this bouncing ball problem will be a future work.

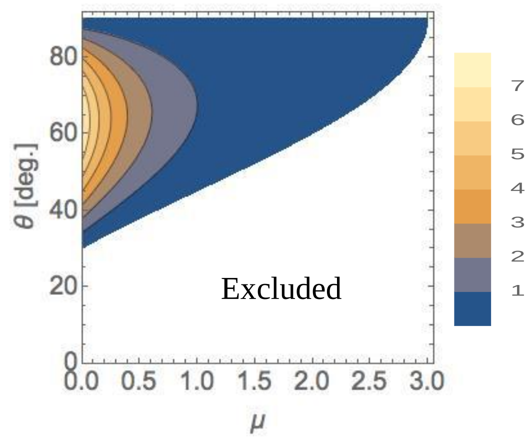

FIGURE 2. Contour plot for $\tilde{L}$. At $(\mu, \theta) = (0, 63.2°)$, $\tilde{L}$ has the maximal value 7.04.

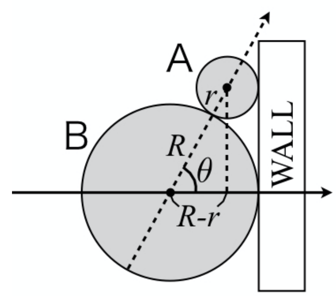

FIGURE 3. Set up to realize the maximal angle.

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