A screw-biased Galton board

EDVCATIO PHYSICORVM

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ISSN 1870-9095

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(Received 5 January 2022, accepted 12 March 2022)

Abstract

We have built a Galton board where the traditional nails can be encapsulated by threaded (and removable) hollow cylinders, eventually obtaining a board with screws. Ideally our board, with 12 rows, should have 13 columns. Due to material limitations, it has only 5 columns, making the contribution of the absorbing side walls significant. The absorption in each of these 5 columns is calculated, in the nail and screw arrangement, making use of the binomial distribution that would be valid for an ideal board. In the screw arrangement, a bias to the right of value $p = 0.53 \pm 0.14$ arises, which increases to $p = 0.58 \pm 0.11$ when the absorption effect is corrected. Using the gambler's ruin theory it is possible to calculate the "average life" of the marble before it is absorbed in its downward motion: it would be 16 shocks for the nails and a little more than 14 shocks for the screws. Although these "average lives" have not been directly measured, our experimental results are consistent with these values.

Keywords: probability, Gaussian distribution, normal distribution, binomial distribution, bias, absorbing walls, quincunx, bean machine, chirality, vortices, screw thread, random walks.

Resumo

Construímos uma tábua de Galton onde os tradicionais pregos podem ser encapsulados por cilindros ocos rosqueados e removíveis, obtendo-se assim uma tábua com parafusos. Idealmente nossa tábua, com 12 linhas, deveria ter 13 colunas. Por limitações materiais, ela tem apenas 5 colunas, tornando significativo a contribuição das paredes laterais absorvedoras. A absorção em cada dessas 5 colunas é calculada, no arranjo de pregos ou parafusos, fazendo-se uso da distribuição binomial que seria válida para uma tábua ideal. No arranjo com parafusos, surge uma distribuição com bias para a direita de valor $p = 0{,}53 \pm 0{,}14$ e que se eleva para de $p = 0{,}58 \pm 0{,}11$, quando se corrije o efeito da absorção. Usando a teoria da ruína do jogador é possível calcular a "vida média" da bolinha de gude antes de ser absorvida no seu movimento descendente: seria de 16 choques para os pregos e um pouco mais que 14 choques para os parafusos. Embora essas "vidas médias" não tenham sido diretamente medidas, nossos resultados experimentais são compatíveis com esses valores.

Palavras-chave: probabilidade, distribuição gaussiana, distribuição normal, distribuição binomial, viés, paredes absorvedoras, quincôncio, máquina de probabilidade, quiralidade, vórtices, rosca de parafuso, passeio aleatório.

I. INTRODUCTION

The normal distribution, represented by a Gaussian curve (bell shape), is of fundamental importance in the analysis of random variations of many phenomena (although there are notable exceptions [1]). The Galton board, formed by an inclined plane through which a marble descends and collides with pins arranged in the form of quincunxes (like the figure of the number 5 on a die), shows this distribution, provided the number of pins is quite large [2]. This distribution of the marbles in the lower collection boxes (called bins) is discrete and, strictly speaking, cannot be represented by the normal distribution, which is continuous. In fact, we have an approximation of the correct distribution, called binomial (see equation 1). This approximation process is a particular case of the central limit theorem [3, 4], which essentially states that the sum of random independent variables tends to a Gaussian distribution when the number of variables approaches infinity.

This apparatus was first built by Sir Francis Galton in 1873 [2] and nowadays can also illustrate phenomena such as nonlinear dynamics [5], granular systems mixing [6], mechanical [7] or light [8] diffusion, ion transport in organic compounds [9], etc.

Our board differs from the traditional one in two basic aspects: the pins that are normally cylindrical (nails) can now be replaced by cylinders hollowed out by helices (screws). This allows inducing a preferential direction for the marbles after each shock (a bias). Another difference is that, due to material limitations, our board is narrower than ideal and the side walls limit the movement of the marbles.

II. MATERIALS AND METHODS

A standard Galton board is wide enough that the side walls do not interfere with the movement of the marbles [2, 10, 11]. The larger the number of rows with pins, the larger the number of bins, which collect the marbles. So, if we have 12 rows, we should have 12+1=13 bins, commonly numbered from left to right (for someone looking at the board) from 0 to 12. These 13 bins would include all possibilities, from a marble suffering the 12 deviations to the left (falling in bin zero) to one with the 12 deviations to the right (falling in bin thirteen). In no case would there be a collision with any side walls of the board.

In this situation, the probability P_k for a marble to fall into bin k, is given by the binomial distribution function [11, 12]:

$$P_k = \frac{n!}{k!(n-k)!}\, p^k (1-p)^{n-k}, \qquad (1)$$

where n is the number of times the experiment (collision) repeats itself and p is the probability of "success". We choose "success" for marble move to the right after the collision with a pin. So-called Bernoulli trials [12] are assumed, that is, three hypotheses are considered valid: the events are independent (it does not depend on what happened before), the possible outcomes are just two (“success” or “failure” or, in our case, right or left) and the probability of a “success” is always constant (but not necessarily equal to the probability of a “failure”).

There are several paths that lead to a given bin. In common, they must always have the same number of left and right turns. The second term on the right-hand side of the *equation 1* (the one that does not involve factorials) gives us the probability of the marble making this specific number of turns, regardless of the order in which they are made. Whereas the first term on the right-hand side of the *equation 1* (the one involving the factorials) tells us how many trajectories there are with that number of turns, each trajectory being defined by the order in which the turns are performed [11].

Our board, shown in *figure 1*, has 12 effective lines of pins (nails or screws), each of which has 5 or 4 equally-spaced pins alternately. Therefore, the arrangement of the pins is in quincunx, that is, just like the figure that represents number 5 of a die. The nails can be encapsuled with steel sleeves, hollowed out by helices in the exterior, becoming then like screws. In the photo appear 13 lines, but the marble does not hit the first one (with 4 pins). This would only happen if it was thrown inclined, which is not the case in our experiment. There are also (only) 5 vertical columns which, at the bottom, will correspond to 5 bins where the marble may end, numbered 4 to 8 from left to right. This numbering, odd at first glance, is chosen to be in accordance with that corresponding to an identical Galton board, but without side walls, where the possible bins would go from 0 to 12 (in both cases number 6 is the central bin). Furthermore, there is an extra space (as if they were half columns) at the extreme right and left sides of our board. The marble, when it reaches these spaces, is imprisoned and cannot continue its journey down towards the 5 bins. When this occurs, for reasons that will be better explained below, the marble is deliberately removed, not contributing to the final count in the bins. It is as if the side walls were absorbing walls.

The measurements of our Galton board are (with errors of the order of ± 5%):

- Board: width = 120mm, length = 330mm, inclination with horizontal = 25 degrees, made of wood.
- Nails: diameter = 3mm, height = 22mm, made of steel.
- Separations: horizontal between consecutive nails = 24mm, vertical between consecutive nails = 24mm.
- Screwed "nail sleeve": height = 25mm, internal diameter = 4mm, external diameter = 6mm, threads per millimeter = 0.75, right thread, made of steel.
- Marble: diameter = 18mm, made of glass, distance between the point where it is dropped and the first pin it hits = 90mm.

In our case the marbles were launched one by one, 800 times for nails and again for screws. In some experiments they are thrown almost simultaneously and collisions among them occur, at least during a transient time [2,6,13]. It becomes clear that the hypothesis of the binomial distribution is undermined, if there are such collisions. Nevertheless, it seems that significant changes from the one-by-one launches method (like ours) would be restricted to the dynamics of the marbles, with different dependencies of their average velocity on certain parameters [6,13].

As explained earlier, the marble, upon touching one of the two side walls, becomes trapped and is removed from our Galton board. This corresponds to absorbing walls, as opposed to reflecting walls, where the marble would continue on its downward journey. Why we choose to remove? The fundamental reason is that those marbles get trapped due to the particular arrangement in our board. Therefore, its relocation would be somewhat arbitrary, as to the modulus and direction of the velocity with which this would be accomplished. Moreover, an experiment with reflecting walls [13] shows that when these walls are very close together there is a tendency to form a uniform final distribution, without a well-defined maximum. This would hinder the visualization of an eventual bias, introduced when the nails are replaced by screws. The bias would cause an asymmetry in the final distribution of the marbles using screws, in relation to the expected symmetrical distribution using nails. However, if, when using nails, but due to the proximity of reflecting walls, the distribution tends to be uniform, when changing to screws, the eventual introduced bias would be masked, since the uniform distribution is always symmetrical. These are the two reasons why we opted for absorbing rather than reflecting walls.

FIGURE 1. Our Galton board with 12 effective lines and 5 columns ending in bins that collect marbles. The choice of numbering for the bins follows the standard of traditional broad boards (not limited by walls). On the left, one can sees the removable sleeve that turns the nails into screws.

III. RESULTS AND DISCUSSION

A. Quantifying the absorption for nails

Experimental distribution of the marbles are shown in *figure 2* for nails and *figure 3* for screws. We can calculate, from *equation 1*, what the binomial distribution would look like (without bias and without sidewalls) in the central 5 bins, if there were 12 bins on the Galton board. If we subtract, from these ideal values, the values obtained in the experiment with nails, we will obtain the absorption A_k, for each of the 5 bins of our nail board. As we are assuming perfect symmetry, the experimental values will be equalized on the left and on the right, with the average between these 2 values. The resulting absorption values for each of the bins are:

- $A_6 = -11\%$,
- $A_5 = A_7 = 1\%$,
- $A_4 = A_8 = 45\%$.

Thus, as expected, the effect of the absorbing walls is greater in the bins closest to these walls (bins 4 and 8). But surprising is that the absorption effect is negative in the central bin 6. Put another way, there is an increase in the number of balls in this bin relative to what would be expected if there were no absorber walls. It is difficult to imagine a plausible mechanism for this negative absorption. What seems likely is that it is an upwards fluctuation from the expected value for this central bin. But this explanation, of statistical fluctuations, fails to explain how absorption decreases without variations in behavior (monotonically) from the bins farther away toward the central one. In short, the only way we can see to explain this negative absorption with absorbing walls compared to the case without walls, are statistical fluctuations, coming from a not sufficiently high sampling (800 balls thrown in the total). But if this were so, it would also be reasonable that the fluctuations in the other bins would result in a non-monotonic absorption behavior, as one moves from the side bins to the central one; but this is not what is experimentally verified.

B. Quantifying the bias for screws

Comparing our experimental distribution for the screw board, an asymmetry is observed with respect to that with nails. The ideal binomial distributions with biases are also asymmetric, and due to the shock of the marbles with the right threaded screws, it can be assumed that each shock will cause a constant bias in the movement of the marbles. Therefore, one can search for the value of the theoretical bias that would lead to a distribution similar to our screw board. But, regardless of the simulated bias we choose [10,11], there are no binomial that resemble our experimental distribution. Among the possible reasons for that, a main issue is that our experiment has absorbing lateral walls and equation 1 does not take this into account. In spite of that, a possibility to evaluate the experimental bias is by averaging all 10 combinations of the 5 bins, two by two. Thus, for example, the relationship between the marbles in the central bin 6 with

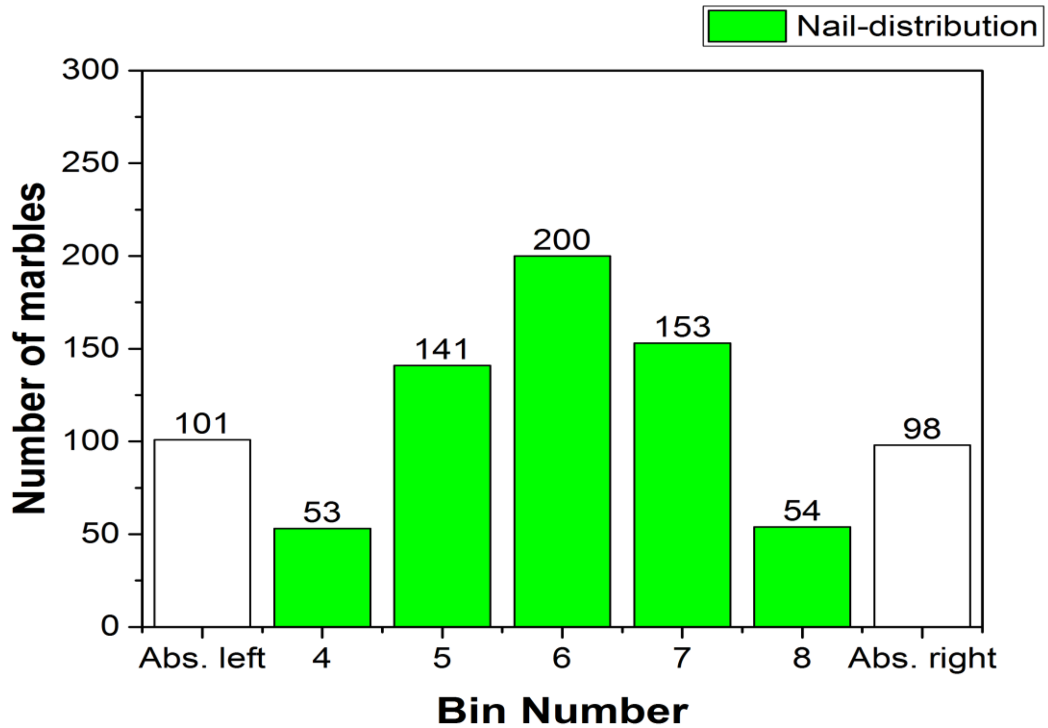

FIGURE 2. The experimental distribution of 800 marbles in the five bins in our Galton board with nails. The colorless columns show absorption on the left and right walls.

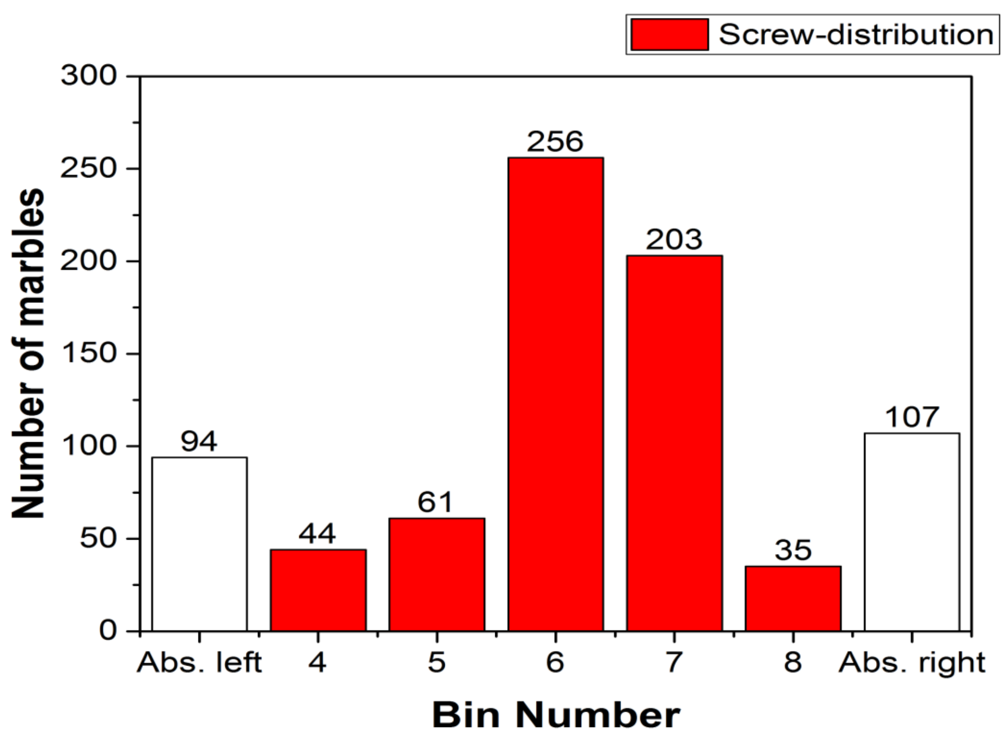

FIGURE 3. The experimental distribution of 800 marbles in the five bins in our Galton board with screws. The colorless columns show absorption on the left and right walls.

their first neighbor on the left (bin 5) worth 256/61 = 4.20. In ideal binomial distributions [10] this corresponds to a 0.75 bias, or p = 0,75 in the binomial formula of equation 1 (remembering that p = 0,50 would mean no bias). Taking all

10 possible bins combinations, we obtained an experimental mean value of $\boldsymbol{p = 0.53 \pm 0.14}$ for our screw board. Note that the standard deviation has two significant figures, which is acceptable, especially when its first significant digit is the unit [14].

On the other hand, it is also possible to evaluate the bias by taking into account that there is absorption in the side walls. One way to do this would be to consider valid the empirical absorption already observed in our nail board. That is, we will assume that the same absorption for each bin of the nail board also applies to the screw board. But this is not enough, because it is important to consider the asymmetry in the absorption of the screw board (which arises in addition to that in the distribution itself). That is, the absorbed marbles on the left and on the right are in a ratio of 94/107 = 0.88. Considering these two terms, the absorption for each bin observed in the nail board, and the left-right asymmetry in absorption, it is possible to introduce a correction in the effective number of marbles in each bin in the screw board. Having done this, it is observed that, again, the distribution does not correspond to any ideal binomial distribution, whatever the bias chosen in the simulation. But, as before, it is possible to evaluate the experimental bias by averaging over all 10 combinations of the 5 bays, two by two. Thus, the new experimental mean bias value for our screw board (considering the lateral absorption) is $\boldsymbol{p = 0.58 \pm 0.11}$.

C. Application of the gambler's ruin theory

The absorbing side walls introduce a difficulty in analytically forcast the final distribution, even in the simplest case of nails, where there is no bias. At any rate, as is physically consistent, the motion cannot continue indefinitely with the marble having a non-zero probability of being absorbed. That is, in the limit where the number of lines tends to infinity, it is easy to predict the final distribution of marbles: it will be uniform, with all null values (all empty bins at the end). No marble will reach the bottom of the board. Besides being a reasonable result, it is possible to demonstrate this mathematically using gambler's ruin theory [15], which we will also use below for another calculation. It is also possible to calculate the "average lifetime" of a marble, that is, how many shocks it will suffer on average before it is absorbed by one of the two walls. This calculation is based on a standard random walk problem known as gambler's ruin [15]. It gets its name because it is related to how many bets on a roulette wheel a player can make before losing his money. Remember that in a casino the probability of the banker winning is always a little higher (there is a bias favoring the house) because of the roulette zeros (1 in Monte Carlo and 2 in American casinos). When the result is "zero", all bets are for the banker. Making an analogy between this problem and our Galton board (nails or screws), suppose that marble movements to the left are failures (bet loss) and moves to the right are successes (bet gain). If from the first shock, in its zig-zag movement, there is a net displacement of 4 times to the left, the marble will touch the left wall and be absorbed. In a similar way, with a net displacement of 4 times to the right, the marble will be absorbed by the right wall. The correspondence would be a player with start-up capital of 4 dollars, who will abandon roulette when he doubles his initial capital or, of course, when he is out of money, "ruined". Therefore, with this analogy, the gambler´s ruin theory predicts that the "average lifetime" of the marble before it is absorbed is [15]:

$$E = \frac{c}{1-2p} - \frac{2c}{(1-2p)} \frac{\left(\frac{1-p}{p}\right)^{c} - 1}{\left(\frac{1-p}{p}\right)^{2c} - 1}. \qquad (2)$$

In the above *equation 2*, E represents the average number of shocks the marble must suffer before being absorbed, c the "start-up capital" and p still is the probability of the marble going to the right. Using this equation, if the bias of our screw board is taken to be $p = 0.58$ (or $p = 0.53$, when correcting for absorption), the marble should most likely be absorbed after $\boldsymbol{E = 14.3}$ (or $\boldsymbol{E = 14.2}$, when correcting for absorption) shocks. That is, a larger number than the 12 shocks that occur in practice. This is consistent with the experimental fact that the percentage of absorbed marbles is smaller than those not absorbed, in the 800 throws.

This same *equation 2* is indeterminate for $p = 0.5$ (no bias). In this case, one should use the following equation [15]:

$$E = c^2. \qquad (3)$$

If we assume that, ideally, the nail board has no bias, then we have precisely this case, of $p = 0.5$. Since $c = 4$, the prediction is an average lifetime of $\boldsymbol{E = 16}$ shocks before absorption. Again, not in contradiction with our experimental results.

IV. CONCLUSIONS

We show that an asymmetric distribution can be obtained by modifying the traditional Galton board by exchanging nails for screws. The exit direction of the marbles, after collisions with the screws, has a bias to the right. It is reasonable to assume that this bias would be to the left if the chirality associated with the screws were reversed and they were left-hand threaded. Galton boards, especially smaller ones like ours, are often used as toys, although various probability phenomena (such as binomial and Gaussian distributions) can be studied with them. Here we can experimentally introduce the presence of bias, by using screws. There are several computer simulators available that demonstrate bias, although for students real experiments are generally more attractive.

The ease of constructing a small board is counterbalanced by the more complex mathematical analysis due to the presence of the side walls. In our case they were absorbing. We were able to measure the absorption value in each column and also the bias associated with the board as a whole. Important parameters to control the numerical value of the bias should be relationships between the radius of the marbles and the thread dimensions. In summary, the use of removable sleeves allows the same quincux obstacle configuration to be easily interchanged between nails (symmetric distribution) and screws (asymmetric distribution), extending the teaching and research applications of traditional Galton boards.

ACKNOWLEDGMENTS

We thanks A. Kocinski and G. G. F. Santos for primary aids and Fundação Araucária for the junior scientific initiation (Pibic Jr.) scholarships.

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