Winding motion in a spiral-like trajectory

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Abstract

In this article I shall describe an easily constructed apparatus for an experiment on winding motion in a spiral-like trajectory in three dimensions. The experimental results show how the total time of the process depends on the initial speed, and the total time has its maximum value of 16.3s for a speed of 2.67m/s. The experimental results were in good agreement with the theoretical predictions. The analytical solution of the problem is original.

Keywords: Winding motion, conservation of energy, angular velocity.

Resumen

En este artículo se describe un aparato de fácil construcción para un experimento sobre el movimiento de aire en una espiral en tres dimensiones. Los resultados experimentales muestran cómo el tiempo total del proceso depende de la velocidad inicial y el tiempo total que tiene su valor máximo de 16,3s para una velocidad de 2,67m/s. Los resultados experimentales se encontran en buena concordancia con las predicciones teóricas. La solución analítica del problema es original.

Palabras clave: Liquidación de movimiento, conservación de la energía, velocidad angular.

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I. INTRODUCTION

Many texts [1, 2, 3] contain conical pendulum whose ball travels in a horizontal circle. The ball is suspended by a string. If a pole suddenly stands upright into the circle, the string winds around the pole until the ball ultimately hits the pole. Let us consider the total time of the process for a initial speed of the ball theoretically. We can guess the total time will be short when the initial speed is very low or very high. Then, let us calculate the initial speed when the total time has its maximum value, and compare it the experimental result. This problem has not been published yet.

II. EXPERIMENTAL PROCEDURE AND RESULTS

Let us assume that the angular velocity of the pole is the same as that of the ball, which is allowed to swing in a horizontal circle and so has a circular path of radius r_0 with a constant speed v_0. If we look at the apparatus from above, we can measure r_0 using a scale (a ruler or similar on the bench below). Then, using the value of r_0, the value of v_0 is given by the formula:

$$v_0 = \sqrt{r_0 g (r_0 - a)} \Big/ \left\{ l_0^2 - (r_0 - a)^2 \right\}^{1/4}. \qquad (1)$$

Here $l_0 = 1.0$ m, $a = 8.0\times10^{-3}$ m and g is the acceleration due to gravity, 9.8 m/s^2. The mass of the ball is 6.4×10^{-2} kg and its diameter is 2.4×10^{-2} m. The length of the pole is about 1.3 m.

In this apparatus only the top of the pole can rotate. A hand - drill can be used to rotate the top using a long metal rod (to which the top is firmly attached) which passes through a tube: the lower end of the rod is held in the chuck of the hand - drill. The tube and the drill are clamped to the edge of the bench to keep the rod upright and to ensure it is able to rotate smoothly. The handle of the drill is turned by hand at a steady rate so that the top of the pole rotates with a constant speed.

If the top stops abruptly, the ball moves almost along a quadrant with the same constant speed of v_0, since a is very small compared with l_0 ($a << l_0$). The ball takes time t_1 to move along the quadrant, and t_1 is a quarter of period of conical pendulum. Then, t_1 is given by the following formula:

$$t_1 = \pi \left\{ l_0^2 - (r_0 - a)^2 \right\}^{1/4} \Big/ 2\sqrt{g} . \cdot \cdot \cdot \cdot (2)$$

After this quarter revolution, the ball travels (during time t_2) in a spiral -like trajectory as the string winds around the pole until the ball ultimately hits the pole. The total time, $t_0 = t_1 + t_2$, is measured by a stopwatch, which can be read accurately to within 0.1 s.

The purpose of this experiment is to show how t_0 depends on the initial speed v_0. The experiment was carried out many times at different initial velocities less than 3.8 m/s (which correspond to the maximum speed required to keep this particular pole from swinging due to tension).

In Figure 2, the circles indicate experimental points. Figure 3 is a stroboscopic photograph for an initial velocity of 1.4 m/s.

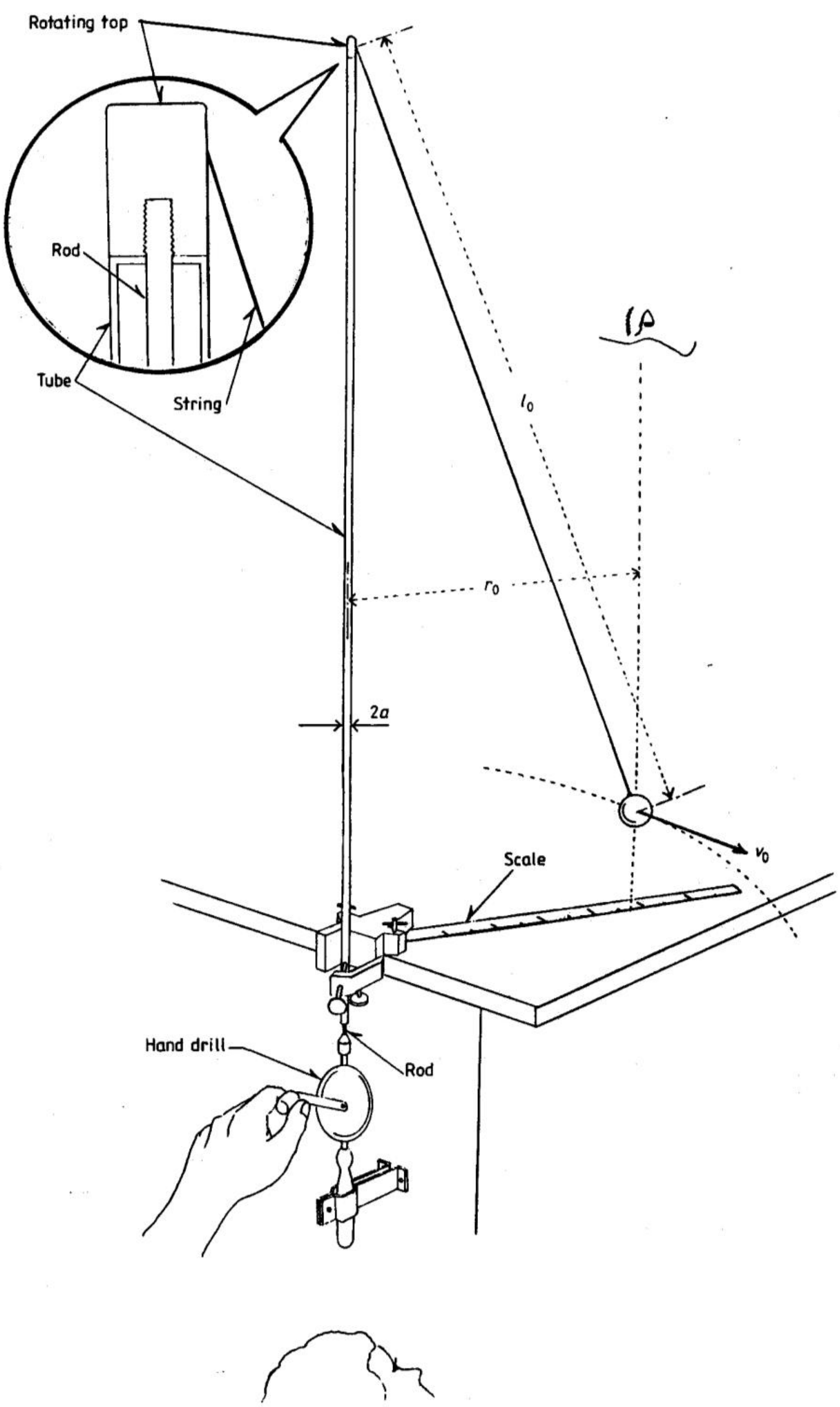

FIGURE 1. The apparatus for measuring the time t_0. Only the top of the pole can rotate.

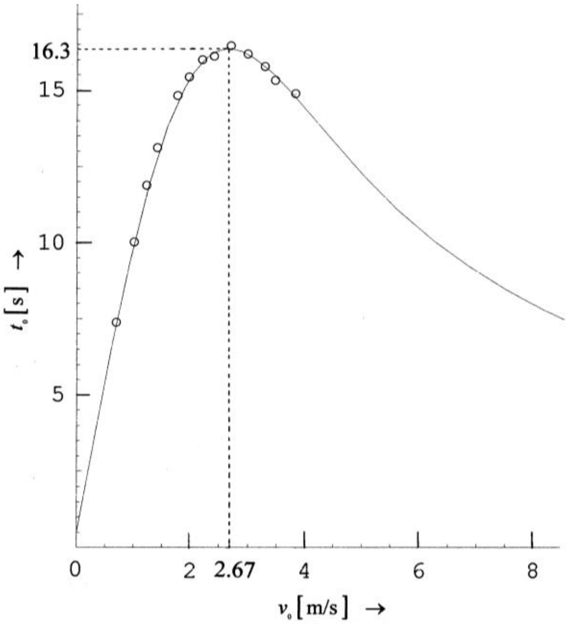

FIGURE 2. t_0 as a function of v_0. The solid line is calculated, and circles (○) are experimental points.

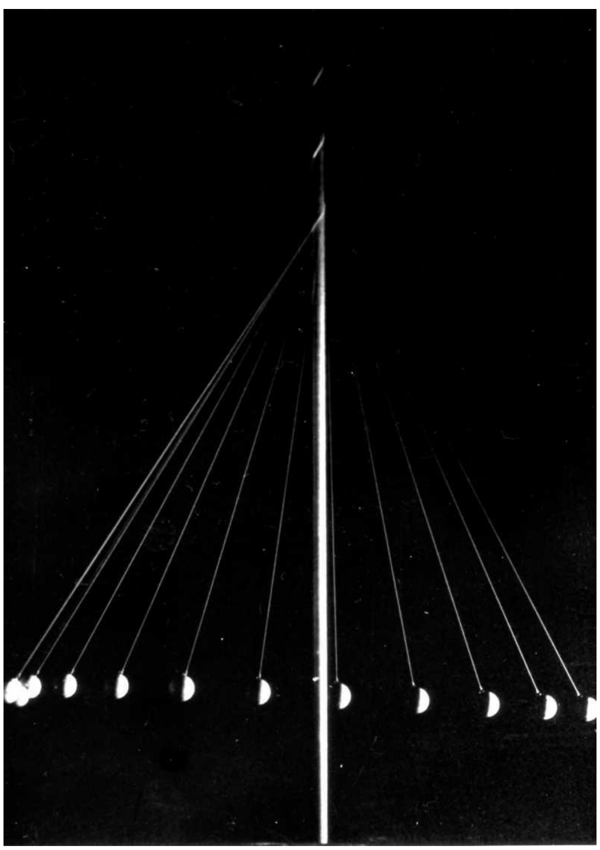

FIGURE 3. Stroboscopic picture of winding motion.

III. THEORY

Let us consider the time t_2 for a given velocity of the ball v_0 theoretically, assuming bulk of the ball, mass of the string and air resistance are negligible. As shown in Fig.4, the string is fastened at a point A.

At time t_1, right after the quarter revolution, the string is tangent to the side of the pole, and it makes an angle φ_0 below horizontal line. At an arbitrary time $t_1 + t$, the position of the ball is B', and the string makes an angle φ with respect to the horizontal. If the point of contact between the string and the pole moves from B to C in a very small interval of time dt, the ball moves from B' to C' in the same time and its incremental change of height is dh; at the same time the angle φ is changed by $d\varphi$. We can then write

$$dh = -l\cos\varphi \cdot d\varphi \quad (\ d\varphi < 0 \ ) . \cdot \cdot \cdot \cdot (3)$$

The pull of the string, T, does not work, since the displacement is perpendicular to T at all times. Hence, using the principle of conservation of energy, $mgh + \frac{1}{2}mv^2 =$ constant, we obtain

$$mg \cdot dh + mv \cdot dv = 0 . \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 4)$$

The instantaneous speed v of the ball is defined as

$$v = \frac{ds}{dt}, \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (5)$$

where ds is the increment of displacement which the ball has during the short time interval dt. This speed can be separated into horizontal and vertical elements represented by $l\cos\varphi \cdot d\theta$ and $-l\cos\varphi \cdot d\varphi$, respectively (see Fig. 5); where θ is the angle $A''OB''$ subtended by arc $A''B''$ (as shown in the plane figure of Fig. 4), and $d\theta$ is the angular displacement in the time interval dt. Hence, $ds = \ell\cos\varphi\sqrt{(d\theta)^2 + (d\varphi)^2}$, and equation (5) can be rewritten as

$$v = l\cos\varphi\sqrt{\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\varphi}{dt}\right)^2} . \cdot \cdot \cdot \cdot \cdot \cdot \cdot (6)$$

From the geometry of the situation (Fig. 4), it is seen that the distance with $-dl$ ($dl < 0$) from B to C is given by the relation

$$-dl = \frac{a \cdot d\theta}{\cos\varphi} \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (7)$$

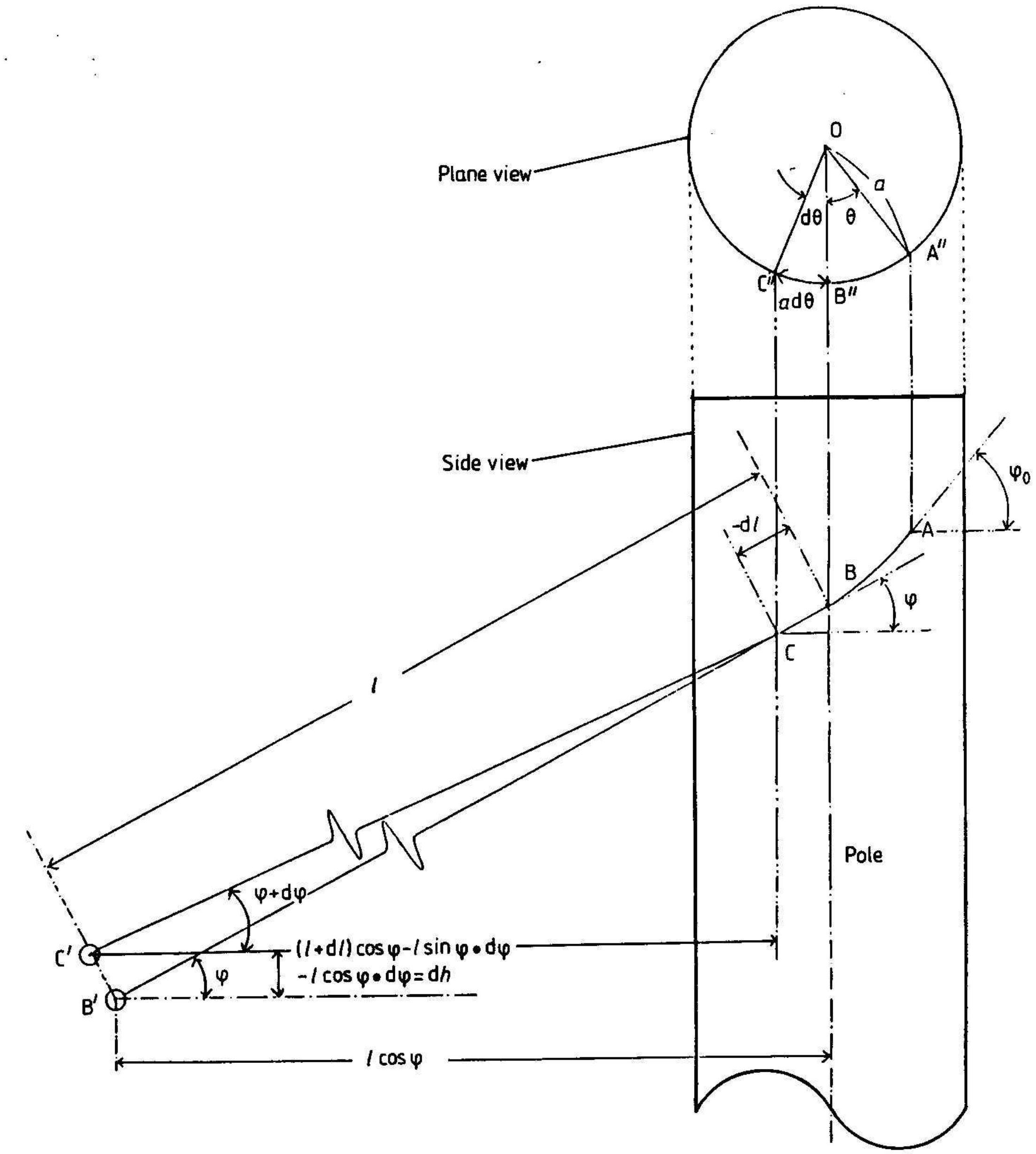

FIGURE 4. The string is fastened at point A, and it leaves the pole at point B and C at time $t_1 + t$ and $t_1 + t + dt$, respectively.

From equations (6) and (7)

$$\frac{v}{\cos\varphi} = \frac{-l\cos\varphi\sqrt{\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\varphi}{dt}\right)^2}}{a\frac{d\theta}{dl}}$$

$$= -\frac{l\cos\varphi}{a}\cdot\frac{dl}{dt}\sqrt{1+\left(\frac{d\varphi}{dt}\right)^2}$$

Since a is very small as compared with l_0, the value of $d\varphi$ will also be very small as compared with the value of $d\theta$. Hence we can neglect the term $(d\varphi / d\theta)^2$. This approximation leads us to the following formula,

$$\frac{v}{\cos\varphi} = -\frac{l\cos\varphi}{a}\cdot\frac{dl}{dt}\cdot\cdot\cdot \qquad \cdot\cdot\cdot\cdot\cdot (8)$$

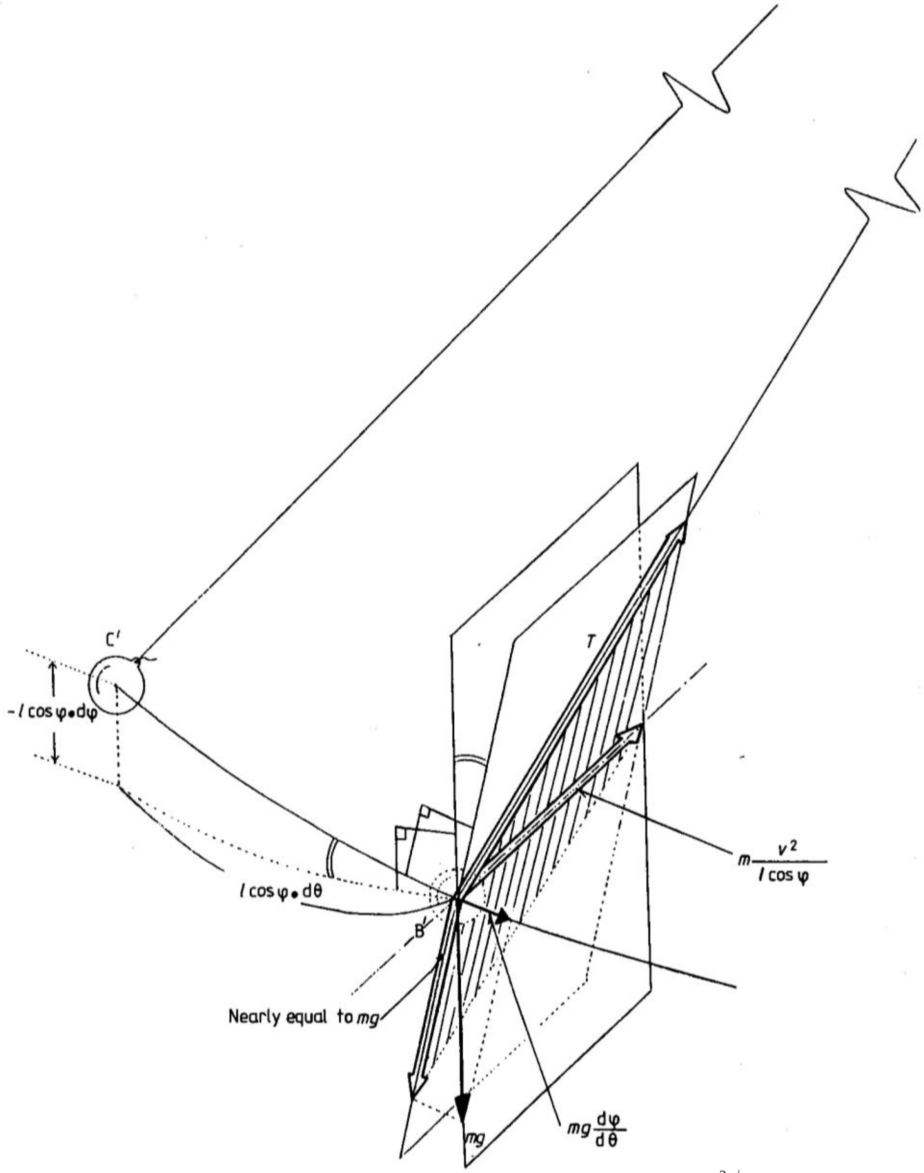

FIGURE 5. The resultant of *m*g and *T* is the centripetal force $mv^2/l\cos\varphi$ approximately.

On the other hand, when the ball is at the point B', the two forces acting through the common point B' are the weight of the ball *m*g and the string tension *T*, as shown in Fig. 5. The resultant of *m*g and *T* is the centripetal force $mv^2/l\cos\varphi$, approximately. Therefore,

$$\tan\varphi = \frac{mg}{mv^2/l\cos\varphi}\,. \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (9)$$

In Eq. (9), we also have neglected a very small force $mg\cdot(\mathrm{d}\varphi/\mathrm{d}\theta)$ that tends to retard the motion. Differentiating Eq. (9), we find

$$\cot\varphi\cdot d\varphi + 2\tan\varphi\cdot d\varphi = \frac{1}{l}dl - 2\frac{1}{v}dv\,. \cdot \cdot \cdot (10)$$

We therefore get the following formula from Eqs. (3), (4), (9) and (10)

$$\frac{1}{l}dl = 4\tan\varphi\cdot d\varphi + \cot\varphi\cdot d\varphi\,. \cdot \cdot \cdot \cdot \cdot (11)$$

Integration of both sides of equation (11) gives

$$l = \frac{l_0\cos^4\varphi_0}{\sin\varphi_0}\cdot\frac{\sin\varphi}{\cos^4\varphi}\,. \cdot \cdot \cdot \cdot \cdot \cdot (12)$$

Eq. (12) gives a relationship between l and φ.

In the case of a true point particle, we find that the final value of φ is zero by setting l = 0 in this formula. By differentiating equation (12) with respect to time $t_1 + t$, we find that

$$\frac{dl}{dt} = \frac{l_0 \cos^4 \varphi_0}{\sin \varphi_0} \cdot \frac{(\cos^5 \varphi + 4 \sin^2 \varphi \cdot \cos^3 \varphi)}{\cos^8 \varphi} \cdot \frac{d\varphi}{dt} \cdot \cdot \cdot (3)$$

On the other hand, from equations (8), (9) and (12), we find

$$\frac{dl}{dt} = -\sqrt{\frac{a^2 \mathrm{g} \sin \varphi_0}{l_0 \cos^4 \varphi_0}} \cdot \frac{\cos \varphi}{\sin \varphi} \cdot \cdot \cdot \cdot \cdot \cdot \cdot (14)$$

Combining equations (13) and (14), we get

$$dt = -\sqrt{\frac{l_0^3 \cos^{12} \varphi_0}{a^2 \mathrm{g} \sin^3 \varphi_0}} \left(\frac{\sin \varphi}{\cos^4 \varphi} + 4 \frac{\sin^3 \varphi}{\cos^6 \varphi} \right) d\varphi . \qquad (15)$$

Integration of both sides of equation (15), and using the fact that at the final time $t_0 = t_1 + t_2$ and the angle $\varphi = 0$, we obtain

$$t_2 = \sqrt{\frac{l_0^3}{a^2 \mathrm{g} \sin^3 \varphi_0}} \left(\frac{1}{5} \cos^6 \varphi_0 - \cos^3 \varphi_0 + \frac{4}{5} \cos \varphi_0 \right) . \qquad (16)$$

Using following equation, $l_0 \sin \varphi_0 = \sqrt{l_0^2 - (r_0 - a)^2}$, we can rewrite equation (2) into

$$t_1 = \frac{\pi}{2} \sqrt{\frac{l_0 \sin \varphi_0}{\mathrm{g}}} \cdot \cdot \cdot \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot (17)$$

Then, the total time t_0 is given by

$$\begin{aligned} t_0 &= t_1 + t_2 \\ &= \frac{\pi}{2} \sqrt{\frac{l_0 \sin \varphi_0}{\mathrm{g}}} + \sqrt{\frac{l_0^3}{a^2 \mathrm{g} \sin^3 \varphi_0}} \cdot \end{aligned}$$
$$\left(\frac{1}{5} \cos^6 \varphi_0 - \cos^3 \varphi_0 + \frac{4}{5} \cos \varphi_0 \right) . \qquad (18)$$

Furthermore from equation (9), the relationship between v_0 and φ_0 is given,

$$v_0 = \sqrt{\frac{l_0 \mathrm{g}}{\tan \varphi_0 ; \sec \varphi_0}} \cdot \cdot \cdot \cdot \cdot \cdot \cdot (19)$$

Using equations (18) and (19), and the computer software “Mathematica”[4], we can calculate numerical values of the time t_0 for different initial velocities v_0. The program is as follows:

$l_0 = 1.0; \ \mathrm{g} = 9.8; \ a = 8.0 \times 10^{-3};$ ↵

$$f[v_0\_] := \frac{\pi}{2} \left(-\frac{v_0^2}{2\mathrm{g}^2} + \frac{1}{2\mathrm{g}} \sqrt{\left(\frac{v_0^2}{\mathrm{g}}\right)^2 + 4 l_0} \right)^{1/2} +$$
$$\left(a^2 \mathrm{g} \left(-\frac{v_0^2}{2\mathrm{g}} + \frac{1}{2} \sqrt{\left(\frac{v_0^2}{\mathrm{g}}\right)^2 + 4 l_0} \right)^3 \right)^{-1/2} *$$
$$(\quad \frac{1}{5} \left(1 - \left(-\frac{v_0^2}{2\mathrm{g}} + \frac{1}{2} \sqrt{\left(\frac{v_0^2}{\mathrm{g}}\right)^2 + 4 l_0} \right)^2 \right)^3 -$$
$$\left(1 - \left(-\frac{v_0^2}{2\mathrm{g}} + \frac{1}{2} \sqrt{\left(\frac{v_0^2}{\mathrm{g}}\right)^2 + 4 l_0} \right)^2 \right)^{3/2} +$$
$$\frac{4}{5} \left(1 - \left(-\frac{v_0^2}{2\mathrm{g}} + \frac{1}{2} \sqrt{\left(\frac{v_0^2}{\mathrm{g}}\right)^2 + 4 l_0} \right)^2 \right)^{1/2} \ ); \hookleftarrow$$

$\mathrm{Plot}[\, f[v_0], \{v_0, 0, 15\}, \mathrm{AxesOrigin} \to \{0, 0\},$
$\mathrm{PlotRange} \to \{0, 18\}]$ ↵

Where $f[v_0]$ is the total time t_0 .

IV. CONCLUSION

In Fig. 2, the solid curve is the calculated curve which is based on equations (18) and (19), and it shows how the time t_0 depends on v_0. We can see from this figure that t_0 has its maximum value of 16.3 s for a speed of 2.67 m/s, and the experimental results were in good agreement with the calculated values. The analytical solution of the problem is original.

In the future, we will construct an apparatus to keep the vertical pole from swinging due to tension at high speed.

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