# A ball rolling on a freely spinning turntable: Insights from a solution in polar coordinates

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**Abstract**

It is well-known that the trajectory of a ball rolling on a freely spinning turntable is a conic section, and that this mechanical problem serves as a proxy for some problems on the motion of accelerated charged particles. This paper describes a direct method to integrate the equations of motion for a ball rolling on top of a freely spinning turntable, without invoking conservation principles as in earlier papers. It is found that the angular speed of the turntable is a simple function of the radial position of the ball only. The explicit expression for the position of the ball in polar coordinates allows a direct analysis of trajectories. The physical parameters leading to open and closed orbits are explicitly identified. Phase diagrams are introduced as a tool for the representation and analysis of trajectories.

**Keywords: R**olling ball, free spinning turntable, phase diagrams for trajectories, Newtonian dynamics, classical mechanics, charged particle motion.

**Resumen**

Es bien sabido que la trayectoria de una bola rodante sobre una tornamesa que pueda girar libremente es una sección cónica; también se sabe que este problema mecánico es análogo, en algunos casos, al movimiento de cargas aceleradas. En este artículo se describe un método directo para integrar las ecuaciones de movimiento de una bola que rueda sobre una tornamesa cuya velocidad de rotación no es fija; a diferencia de otros trabajos anteriores por otros autores, aquí no se invocan principios de conservación. Se encuentra que la rapidez angular de la tornamesa solamente es función de la posición radial de la bola. La ecuación que representa explícitamente la posición de la bola en coordenadas polares permite adelantar un análisis directo de su trayectoria. Se identifican así, y de manera explícita, los parámetros físicos que generan órbitas abiertas y cerradas de la bola. Para la representación y análisis de dichas trayectorias se utilizan diagramas de fase, herramienta que es de uso corriente en otras áreas.

**Palabras clave: B**ola rodante, tornamesa con velocidad de rotación libremente variable, diagramas de fase para órbitas, dinámica Newtoniana, mecánica clásica, movimiento de partículas cargadas.

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## I. INTRODUCTION

According to Romer [1], the earliest formulation of the problem of a marble rolling on a horizontal revolving plane is due to Earnshaw in 1844, and the first mention of a tilted plane is in Routh's treatise on the dynamics of rigid bodies in 1868 (problems section). Routh's solution predicts a counterintuitive trajectory: the ball may move along re-entrant orbits for some values of the initial conditions. When the horizontal disk rotates at constant speed, there may appear circular trajectories, and when the revolving plane is tilted the trajectories are circles drifting along a direction perpendicular to the direction of tilting [2]. Additional interest arises because this problem is a mechanical analog for the motion of a charged particle in orthogonal electric and magnetic fields [3]. The anecdotal fact that Einstein did not want to tackle the latter problem led to a minor skirmish [3, 4]. Perturbation and other methods to solve the problem have been amply discussed since 1980 [5].

A variation of the problem is to let the revolving plane change its rotational speed as a result of the motion of the rolling ball. To show that the trajectory of the ball in this more general case is a conic section, Weckesser [6] integrated the equations of motion invoking conservation of three quantities: total energy, angular momentum about the Z-axis, and the Z-component of the angular momentum of the ball. Rodriguez [7] devised a simpler procedure avoiding conservation of total energy, but using a special orientation of the inertial coordinate system.

We are currently studying in some detail the dynamics of a ball-borne pendulum, sensitive enough to react to the gravitational field of the sun and the moon. The support of the pendulum is a ball rolling on a horizontal plane fixed in the laboratory, hence slowly rotating with the earth. The pendulum exerts a time dependent force on the upper hemisphere of the rolling ball. Since this force is modulated by external gravitational fields, some of the conserved quantities used by Weckesser [6] may not be appropriate for the ball-borne pendulum. The present paper reports the procedure we used to solve Weckesser problem in polar coordinates, without invoking conservation principles. The case of a ball subject to external force is deferred to a future paper.

To make this paper self-contained, section 2 summarizes the formulation of the problem by Weckesser [6], followed by the direct method we used to obtain the turntable angular speed, the resulting expression being simpler than in the earlier papers [6, 7]. Section 3 describes a straightforward procedure to obtain the position of the ball in polar coordinates. Both Weckesser's solution [6] and Rodriguez's solution [7] may be recovered from our general solution. It is found that the special orientation proposed by Rodriguez amounts to a rotation of the system of coordinates so that the ellipse axes are parallel to the rotated coordinate axes. Section 4 illustrates the use of phase diagrams for the analysis of the trajectories, and section 5 closes this note.

## II. THE ANGULAR SPEED OF THE TURN-TABLE

Consider a rotating disk of mass *M* and radius *R* contained in the X-Y plane of a right-handed arbitrarily oriented Cartesian system of coordinates at rest in the laboratory. The origin of the system is at the center of the disk, and the Z-axis is perpendicular to the disk, the unit vectors respectively are **i**, **j**, **k** (see Fig. 1). The turntable is free to rotate at variable angular speed $\Omega(t)$ at time t, the initial speed at time t = 0 is $\Omega_0$, and the instant position of the point of contact of the ball is $\mathbf{r}(t) = x\mathbf{i}+y\mathbf{j}$. A ball of mass *m* and radius *a* is released on top of the disk in rolling mode at time t = 0 from position $\mathbf{r}_0 = X\mathbf{i} + Y\mathbf{j}$ with center of mass velocity $\mathbf{V}_0 = U\mathbf{i} + V\mathbf{j}$. Following Weckesser [6] all variables and parameters are written in non-dimensional form, so that all lengths —like **r**(t), $\mathbf{r}_0$, X, and Y—are in units of *a*, angular velocity is in units of $\Omega_0$, linear velocity and speed are in units oφ $\alpha\Omega_0$, time is in units of $1/\Omega_0$, and force in units of $m\alpha\Omega_0^2$.

Since the weight of the ball is cancelled by the vertical component of the contact force, the net horizontal force of the turntable upon the ball at the point of contact is $\mathbf{f} = f_X\mathbf{i} + f_Y\mathbf{j}$. The two equations of motion for the translation of the center of mass of the ball, and the torque of friction around the center of the ball respectively are

$$\mathbf{r}'' = \mathbf{f}, \qquad (1)$$

$$\beta\boldsymbol{\omega}' = -\mathbf{k}\times\mathbf{f}, \qquad (2)$$

where the angular velocity of the ball is ω = ωΞι + ωΨφ + ωΖκ , Iβ is the moment of inertia of the ball, and the corresponding non-dimensional parameter is $\beta = I_b/ma^2 = 2/5$. The primes denote derivatives with respect to the non-dimensional time. Assuming that the ball always is in pure rolling motion, the velocity of the surface of the ball at the point of contact must equal the velocity of the turntable at the same point, so that

$$\mathbf{r}' + \mathbf{k}\times\boldsymbol{\omega} = \Omega\mathbf{k}\times\mathbf{r}. \qquad (3)$$

Finally, the equation of motion for the torque produced by force **f** upon the turntable is

$$\delta\Omega'\mathbf{k} = -\mathbf{r}\times\mathbf{f}, \qquad (4)$$

where $I_d$ is the moment of inertia of the turntable around the Z-axis, and the nondimensional moment is δ =Iδ/μα2. Previous Eqs. (1)-(4) correspond to Eqs. (5)-(8) in Weckesser's paper.

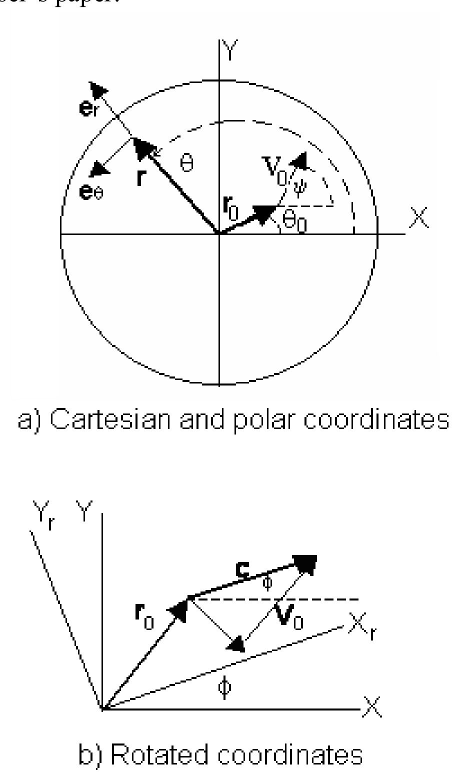


**FIGURE 1**. a) A turntable of radius R rotates with speed Ω the counterclockwise rotation is positive. An arbitrary Cartesian system of coordinates is fixed in the laboratory, the Z-axis runs out of page. Unit vectors $(\mathbf{e}_r,\mathbf{e}_\theta)$ for polar coordinates are shown. In polar coordinates, the position of the point of contact of the ball at times t and t = 0 respectively are (**r**,θ) and $(\mathbf{r}_0, \theta_0)$. b) In the rotated system of Cartesian coordinates $(X_r, Y_r)$ the $X_r$-axis is parallel to the direction of vector **c**, defined by initial position $\mathbf{r}_0 = X\mathbf{i} + Y\mathbf{j}$ and velocity $\mathbf{V}_0 = U\mathbf{i} + V\mathbf{j}$ (see Eq. 6).

Eliminating **f** from Eqs. (1)-(2), integrating and invoking the non-slip constraint Eq. (3),

$$\mathbf{r}' = \alpha\Omega\mathbf{k}\times\mathbf{r} + \mathbf{c}, \qquad (5)$$

where $\alpha = \dfrac{I_b}{I_b + ma^2} = \dfrac{2}{7}$, and the integration constant vector **c** shown in figure 1 is a measure of the initial velocity of the ball relative to the disk,

$$\mathbf{c} = \mathbf{V}_0 - \alpha\mathbf{k}\times\mathbf{r}_0 = c_1\mathbf{i} + c_2\mathbf{j} = (U + \alpha Y)\mathbf{i} + (V - \alpha X)\mathbf{j}. \quad (6)$$

In polar coordinates the orientation of **c** is given by angle $\phi$ defined by $\tan\phi = \dfrac{V - \alpha X}{U + \alpha Y}$. It turns out that the magnitude of **c** is the maximum radial speed that may be attained by the ball when the trajectories are closed, see section 3, paragraph underneath Eq. (17).

Eliminating force **f** from Eq. (4) with the help of all previous equations, Weckesser [6] obtained his Eq. (12), namely

$$\Omega' = \frac{-\alpha(\mathbf{r}\cdot\mathbf{c})}{\delta + \alpha(\mathbf{r}\cdot\mathbf{r})}\Omega. \quad (7)$$

Then, the problem reduces to obtaining **r**(t) and Ω(t) from the system of two coupled differential equations (5) and (7) with the initial conditions t = 0, $\mathbf{r}(0) = \mathbf{r}_0$, $\mathbf{r}'(0) = \mathbf{V}_0$, and Ω(0) = 1. To solve this system conservation principles were invoked both by Weckesser [6] and Rodriguez [7].

Instead, we will integrate directly Eq. (5) by using the standard approach of converting the differential equation into a total derivative. For this we take the dot product of vector **r** with both sides of Eq. (5) to obtain

$$\mathbf{r}\cdot\mathbf{r}' = \alpha\Omega\mathbf{r}\cdot(\mathbf{k}\times\mathbf{r}) + \mathbf{r}\cdot\mathbf{c}. \quad (8)$$

The left hand side of Eq. (8) may be immediately recognized as a total derivative, while the first term on the right-hand side is zero because the vectors are mutually perpendicular. Hence,

$$\frac{1}{2}\frac{d(\mathbf{r}\cdot\mathbf{r})}{dt} = \mathbf{r}\cdot\mathbf{c}. \quad (9)$$

Substituting (9) into (7),

$$-2\frac{\Omega'}{\Omega} = \frac{\alpha}{\delta + \alpha(\mathbf{r}\cdot\mathbf{r})}\frac{d(\mathbf{r}\cdot\mathbf{r})}{dt}, \quad (10)$$

which immediately yields

$$\Omega^2 = \frac{\delta + \alpha(\mathbf{r}_0\cdot\mathbf{r}_0)}{\delta + \alpha(\mathbf{r}\cdot\mathbf{r})} = \frac{\delta + \alpha r_0^2}{\delta + \alpha r^2}. \quad (11)$$

It is noteworthy that the angular speed of the rotating table only depends upon the instant radial position of the ball and the ball's initial radial position, fact that was not recognized in the earlier papers. This means that, in general, the turntable may rotate between a minimum speed, that tends to zero when the ball is far from the rotation axis, to a maximum speed when the ball crosses the origin given by

$$\Omega_{\max}^2 = 1 + \frac{\alpha r_0^2}{\delta}. \quad (12)$$

Whether the disk will eventually attain any one of these extreme values entirely depends of the particular trajectory of the ball, wholly determined by the set of initial conditions of the ball (see next sections).

The simple expression (11) above may be contrasted with the more cumbersome Eq. (13) in the original article [6]

$$\Omega = \frac{L - \mathbf{k}\cdot(\mathbf{r}\times\mathbf{c})}{\delta + \alpha(\mathbf{r}\cdot\mathbf{r})}, \quad (13)$$

and with Eq. (5) in Rodriguez note [7]

$$\Omega = \frac{L + cy}{\delta + \alpha(\mathbf{r}\cdot\mathbf{r})}. \quad (14)$$

In both equations (13) and (14) the numerator also depends of the variable position of the ball (**r** = $x$**i** + $y$**j**), which hides the simple structure of (11). This dependence may be removed in the case of Rodriguez solution by substituting his final Eq. (8),

$$(L + cy)^2 = k(\delta + \alpha(\mathbf{r}\cdot\mathbf{r})),$$

into (14) above. However, Weckesser [6] did not provide an explicit expression for **r**.

## III. THE TRAJECTORY OF THE BALL

Let us turn now to the explicit calculation of the trajectory of the ball, $r$ versus θ. Substituting Eq. (11) into (5),

$$\mathbf{r}' = \alpha\sqrt{\frac{\delta + \alpha(\mathbf{r}_0\cdot\mathbf{r}_0)}{\delta + \alpha(\mathbf{r}\cdot\mathbf{r})}}\mathbf{k}\times\mathbf{r} + \mathbf{c}, \quad (15)$$

which may be integrated in Cartesian coordinates using the procedure explained by Rodriguez [7], but taking into account that in the present paper it is not assumed that vector **c** coincides with the X-axis. Instead, in this paper the integration is carried out in polar coordinates.

At arbitrary time $t$ let θ be the orientation of $\mathbf{r} = r\mathbf{e}_r$ . At t = 0, the constant $\mathbf{c} = c\hat{\mathbf{c}}$ is oriented in the direction ϕ, where the caret represents the unit vector. Decomposing **c** along the instant unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$, and substituting into (15),

$$\mathbf{r}' = \left( \alpha r \sqrt{\frac{\delta + \alpha r_0^2}{\delta + \alpha r^2}} \right) \mathbf{e}_\theta + c\cos(\theta - \varphi)\ \mathbf{e}_r - c\sin(\theta - \varphi)\ \mathbf{e}_\theta . \quad (16)$$

The components of (16) along the unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ respectively are

$$r' = c\cos(\theta - \varphi), \quad (17)$$

$$r\theta' = \alpha r \sqrt{\frac{\delta + \alpha r_0^2}{\delta + \alpha r^2}} - c\sin(\theta - \varphi). \quad (18)$$

Eq. (17) implies that, in the polar coordinates attached to the axis of rotation of the turntable, the magnitude of radial speed varies between zero, when $\theta - \phi = \pi/2$, and $c$, when $\theta - \phi = 0$. The maximum $c$ is attained when the ball crosses the rotated $X_r$ axis in Fig. 2. This means that the maximum radial speed of the ball entirely depends of the initial speed of the ball relative to the turntable, fact that has not been noted before. The direction of **c** also plays a relevant role, as seen below.

To solve the coupled pair of Eqs. (17) and (18), let us introduce the auxiliary variable $G = \sqrt{\delta + \alpha r^2}$ and substitute into (18) to get

$$r\theta' = \frac{G_0}{r'}\frac{dG}{dt} - c\sin(\theta - \varphi). \quad (19)$$

Rearranging (19) and invoking (17) as needed we get

$$rr'\theta' + cr'\sin(\theta - \varphi) = c\frac{d(r\sin(\theta - \varphi))}{dt} = G_0\frac{dG}{dt}. \quad (20)$$

Integrating from $t = 0$ to $t$,

$$G = G_0 + \frac{c}{G_0}\left[ r\sin(\theta - \varphi) - r_0\sin(\theta_0 - \varphi) \right]. \quad (21)$$

Substituting the explicit value of G back into (19) and squaring

$$\delta + \alpha r^2 = \frac{\left[ L + cr\sin(\theta - \varphi) \right]^2}{\delta + \alpha r_0^2}, \quad (22)$$

where $L$ is a numerical constant defined as

$$L = \delta + \alpha r_0^2 - cr_0\sin(\theta_0 - \varphi). \quad (23)$$

Clearly, $L$ depends of initial conditions, and represents the projection on the Z-axis of the angular momentum of the disk plus the orbital angular momentum of the ball $\ell_z$. Note that $L$ may be positive, zero, or negative. As shown by Weckesser [6, p. 737], $L$ is a constant of the motion, that may be also expressed as

$$L = \delta\Omega + \ell_z = \delta\Omega + \mathbf{k}\cdot(\mathbf{r}\times\mathbf{r}') = \delta + \mathbf{k}\cdot(\mathbf{r}_0\times\mathbf{V}_0) = \delta + \ell_0, \quad (24)$$

Where $\ell_0$ is the initial orbital angular momentum of the ball at $t = 0$.

Eq. (22) is a conic section, and represents the general solution for the motion of the ball in polar coordinates with center at the rotation axis. It may be immediately reduced to the forms given by Weckesser [6] and Rodriguez [7]. For the latter define $\theta_r = \theta - \varphi$ and $y_r = r\sin\theta_r$ thus recovering Eq. 8 in Rodriguez paper. Geometrically, the redefinition of angles amounts to a rotation of coordinates through angle $\phi$, defined by the direction of constant **c**, the initial relative velocity of the ball.

On the other hand, Weckesser's solution is in Cartesian coordinates using $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$. Substituting in (22) we get

$$Ax^2 + By^2 + 2c_1c_2xy + 2Lc_2x - 2Lc_1y = F, \quad (25)$$

with

$$A = \alpha(\delta + \alpha r_0^2) - c_2^2,\ B = \alpha(\delta + \alpha r_0^2) - c_1^2,\ F = L^2 - \delta(\delta + \alpha r_0^2), \quad (26)$$

which is the same equation (15) in Weckesser's paper [6] (note that B above is denoted C in the original paper). Also note that our constants A, B and F are somewhat simpler than in the original paper.

The general analysis of the conic curves is simpler in the rotated system of Cartesian coordinates used by Rodriguez, where $r^2 = x_r^2 + y_r^2$, $x_r = r\cos\theta_r$, and $y_r = r\sin\theta_r$. Substituting in (22),

$$A_r x_r^2 + B_r y_r^2 - 2Lcy_r = F_r, \quad (27)$$

where

$$A_r = \alpha(\delta + \alpha r_0^2),\quad B_r = \alpha(\delta + \alpha r_0^2) - c^2,\quad F_r = F. \quad (28)$$

Since $A_r > 0$ it follows from analytic geometry, see for instance Lehmann [8], that the sign of $B_r$ defines the type of conic section: an ellipse if $B_r > 0$, a parabola when $B_r = 0$, and a hyperbola if $B_r < 0$. For $B_r \neq 0$, Eq. (27) may be written in the standard form

$$\pm\frac{x_r^2}{b^2}+\frac{(y_r-Y_C)^2}{a^2}=1, \qquad (29)$$

where as usual *a, b* are the semi-major and semi-minor axes, and the center of the curve is at position $X_C=0, Y_C$ in the rotated system of coordinates, given by

$$a^2=\frac{F_rB_r+L^2c^2}{B_r^2}, b^2=\pm\frac{F_rB_r+L^2c^2}{A_rB_r}, Y_C=\frac{Lc}{B_r}, \qquad (30)$$

and the positive (negative) sign in Eqs. (29) and (30) corresponds to the ellipse (hyperbola). The signs in these equations are correct because it may be easily checked that the numerator is positive

$$F_rB_r+L^2c^2=(\delta+\alpha r_0^2)(\alpha\ell_0^2+\delta V_0^2)\geq 0, \qquad (31)$$

the equality holding for $V_0=0$. As shown in figure 2, ellipses have the semi-major axis *a* parallel to the $Y_r$-axis, and the semi-minor axis *b* parallel to the $X_r$-axis. The parabolic and hyperbolic trajectories have a similar orientation.

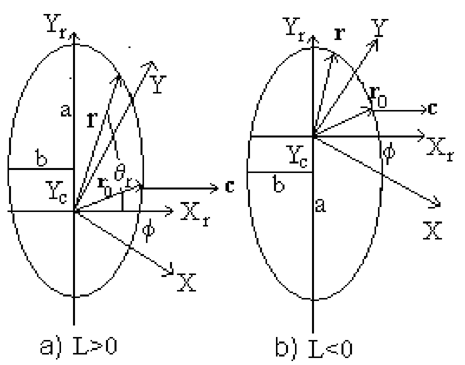


**FIGURE 2**. Parameters and orientation of elliptical trajectories. The $X_r$ axis coincides with the direction of vector **c** defined by the initial conditions. a) Elliptical trajectory for $L>0$. b) Elliptical trajectory for $L<0$.

## IV. PHASE DIAGRAMS AS A TOOL FOR THE ANALYSIS OF TRAJECTORIES

Turning now to the physical interpretation of the condition for closed trajectories $B_r>0$. Substituting $B_r$ from Eq. (28) and *c* from Eq. (6) into inequality (31) we get

$$\ell_0=XV-YU>\frac{V_0^2}{2\alpha}-\frac{\delta}{2}. \qquad (32)$$

Inequality (32) is surprisingly simple. It means that closed trajectories require the ball's orbital angular momentum about the Z-axis to be larger than the translation kinetic energy of the ball $K=V_0^2/2$. Borrowing the technique of phase diagrams from physical chemistry, it is possible to obtain a pictorial representation of inequality (32) that depicts the regions for open and closed trajectories in the K-$\lambda_0$ phase diagram (see Fig. 3). Alternatively, by invoking Eq. (24), inequality (32) may be cast in terms of *L* as

$$L>\frac{V_0^2}{2\alpha}+\frac{\delta}{2}. \qquad (33)$$

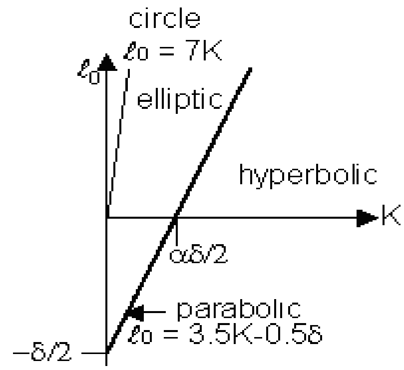


**FIGURE 3.** The K-λ0 phase diagram for the ball trajectories. Elliptical and circular trajectories are generated when the parameters are above the straight bold line. Hyperbolic trajectories appear when the ball has initial parameters below the line. Parameters on the line lead to parabolic trajectories (see inequality 32 in the text). Circles appear when parameters are on the straight line with slope 7.

In expressions (32) and (33) the initial speed appears on both sides of the inequality. Using polar coordinates, inequality (32) becomes

$$r_0V_0\sin\Delta>\frac{V_0^2}{2\alpha}-\frac{\delta}{2}, \qquad (34)$$

where $\Delta=\psi-\theta_0$ is the angle between vectors $\mathbf{r}_0$ and $\mathbf{V}_0$ (recall Fig. 1), $\Delta$ is positive in the counterclockwise direction. Inequality (34) leads to

$$r_0>F(V_0)=\frac{1}{2\alpha\sin\Delta}\left(V_0-\frac{\alpha\delta}{V_0}\right), \text{if } 0<\Delta<\pi,$$
$$r_0<F(V_0)=\frac{1}{2\alpha\sin\Delta}\left(V_0-\frac{\alpha\delta}{V_0}\right), \text{if } -\pi<\Delta<0, \qquad (35)$$
$$V_0<\sqrt{\alpha\delta}, \text{if } \Delta=-\pi,0,\pi.$$

As shown in Fig. 4, inequalities (35) may be represented in a $V_0$-$r_0$ diagram, with angle $\Delta$ as parameter. Of course, Figs. 3 and 4 are representations of the same problem, but they convey slightly different information, as illustrated with the following examples.

*Example 1. Null initial orbital angular momentum*. The condition $\ell_0 = 0$ occurs in three cases:

(a) Ball released at the origin X=0,Y=0 with arbitrary velocity $V_0 \neq 0$, any direction (this is a generalization of example 2 in Weckesser's paper). This case is represented by the horizontal axis in both Figs. 3 and 4. When $V_0 < \sqrt{\alpha\delta}$ the trajectories are elliptical with

$$\frac{a^2}{b^2} = \frac{\alpha\delta}{\alpha\delta - V_0^2}. \quad (36)$$

This means that ellipses are almost circular for small speeds and tends to be very elongated along the $Y_r$-axis when $V_0$ tends to its upper limit.

(b) Ball released at any position X,Y without translational motion $V_0 = 0$. From an experimental viewpoint, this seems to be the easiest condition to attain [1]. This case corresponds to the origin in Fig. 3, and to points along the vertical axis in Fig. 4. The trajectories are degenerate ellipses with $a = b = 0$ (see Eqs. (30) and (31)); hence, the ball remains at rest relative to the laboratory at the initial position X,Y (this is example 3 in Weckesser's paper).

(c) Ball released at any position X,Y with its center of mass moving along a radius. This is a treacherous condition leading to motion that is counterintuitive. As in case (a) this problem is represented by the horizontal axis in Fig. 3, but corresponds to the whole central diagram in Fig. 4. Trajectories are elliptical (but not circular) if $V_0 < \sqrt{\alpha\delta}$, with axes in the ratio

$$\frac{a^2}{b^2} = \frac{\alpha(\delta + \alpha r_0^2)}{\alpha\delta - V_0^2}. \quad (37)$$

This means that ellipses are almost circular when the ball is released at a low speed near the origin and tend to be very elongated along the $Y_r$-axis when the ball is released far from the disk axis and/or when $V_0$ tends to its upper limit.

Similar analyses may be carried out for $\ell_0 \neq 0$. In those cases the line $\ell_0 = const$ is horizontal in figure 3, while the expression $\ell_0 = r_0 V_0 \sin\Delta = const$ leads to a family of curves $r_0 = const/(V_0 \sin\Delta)$ in the appropriate phase diagram of Fig. 4.

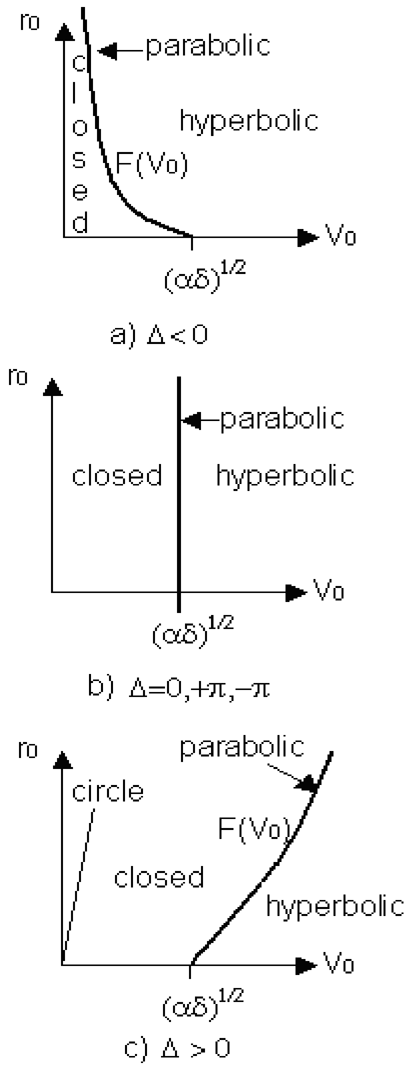


**FIGURE 4.** The $V_0$-$r_0$ phase diagram for ball trajectories. Closed (open) trajectories are generated when the parameters are to the left (right) of the $F(V_0)$ curve. Parameters on the dividing curve lead to parabolic trajectories (see inequalities 35 in the text). Three cases are shown according to the value of angle Δ. a) Negative Δ. b) Δ= 0, or 180º. c) Positive Δ. Circular trajectories appear for parameters on the straight line $r_0 = 3.5V_0$ with $\Delta = \pi/2$.

*Example 2. Circular motion*. From Eq. (30), when $c = 0$ the ellipses become circles of radius $a = b = r_0$ with center at the origin, while the turntable rotates with constant angular speed $\Omega_0$ (see Eq. (11)). Of course, in circular trajectories there is no radial speed, which is consistent with the interpretation of *c* advanced after Eq. (17). Conversely, if $c = 0$ the maximum radial speed must be zero, and the trajectory must necessarily be circular. From the definition of *c* (Eq. (6)) circular motion requires the initial velocity of the ball to be in the same direction as the motion of the disk at the point of initial contact, that is, to be perpendicular to $\mathbf{r}_0$; therefore, $\Delta = \pi/2$. The components of **c** are then

$$c_1 = 0 = U + \alpha Y, c_2 = 0 = V - \alpha X \Rightarrow V_0^2 = U^2 + V^2 = \alpha^2 r_0^2 \quad (38)$$

Hence, in circular trajectories the point of release and the speed of the ball are related by

$$r_0 = \frac{V_0}{\alpha} = \frac{7}{2}V_0. \qquad (39)$$

Eq. (39) is the locus of circular trajectories as shown in the rightmost phase diagram in Fig. 4. Note that the line given by (39) can never cross $F(V_0)$ because the slope of the latter tends to 7/4 when $V_0 \to \infty$ (see Eq. 35a with $\Delta = \pi/2$).

Likewise, the initial orbital angular momentum of the ball is

$$\ell_0 = \alpha r_0^2 = V_0^2/\alpha = 7K. \qquad (40)$$

Since in circular motion radius and speed are constant $\ell_z = \ell_0$ is a constant of the motion. The locus for circular motion given by Eq. (40) is shown in Fig. 3. Previous remarks bring new physical insights to the second part of Weckesser's example 1.

*Example 3. Arbitrary initial speed*. Consider now the case of a ball released with arbitrary speed $V_0$ on the disk. In the phase diagram of Fig. 3, this is a vertical line at $K = V_0^2/2$ that crosses all regions of closed and open trajectories, including the circular case. In this way one may identify the values of $\ell_0$ leading to a particular class of trajectories, say closed. Finally from $\ell_0 = r_0 V_0 \sin\Delta$ one obtains the family of release positions and direction of the velocity compatible with the required trajectory.

Note that there exist external boundaries on the phase diagrams that we have not discussed. These limitations arise from the finite size R of the turntable, and from the coefficient of friction that controls the maximum value that the contact force may take, which implies that $V_0$ cannot be arbitrarily large.

## V. CONCLUDING REMARKS

The present paper described a straightforward method to integrate the equations of motion without invoking additional conservation principles, or limiting the solution to special cases only, as done in previous papers. For the most general case we obtained closed and simple expressions for the angular speed of the turntable and the trajectory of the ball: a conic section in polar coordinates. The initial conditions leading to open and closed trajectories of the ball were obtained. The paper also illustrated the method of phase diagrams to identify the trajectories generated by a set of initial conditions: velocity of the center of mass of the ball and position of release. It is rather surprising that the initial spin of the ball plays no explicit role whatsoever in constraining the motion.

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