Intrinsic gyroradius in the Kaluza-Klein theory

Adrián G. Cornejo
Electronics and Communications Engineering from Universidad Iberoamericana
Calle Santa Rosa 719, CP 76138, Col. Santa Mónica, Querétaro, Querétaro, Mexico.

E-mail: adriang.cornejo@gmail.com

(Received 29 October 2012; accepted 5 February 2013)

Abstract
In this paper, gyroradius for a pointlike charged particle is derived from the Kaluza-Klein theory by introducing a magnetic field. It shows that hypothetic Klein's pointlike particle in the five dimensions spacetime can be circumgyrating in a magnetic field to cover the volume that forms a determined particle in three dimensions. Some analogies of this derivation with the considered topology in the String Theory and M-Theory are commented.

Keywords: Kaluza-Klein theory, Maxwell stress tensor, Gyroradius.

Resumen
En este trabajo, el giroradio para una partícula puntual cargada es derivado de la teoría de Kaluza-Klein por la introducción un campo magnético. Esto muestra que la hipotética partícula puntual de Klein en cinco dimensiones de espacio-tiempo puede estar circumgirando en un campo magnético pata cubrir el volumen que forma una determinada partícula en tres dimensiones. Algunas analogías de esta derivación con la topología considerada en la Teoría de Cuerda y la Teoría-M son comentadas.

Palabras clave: Teoría de Kaluza-Klein, Tensor de presión de Maxwell, Giro-radio.

PACS: 04.40.Nr, 04.50.-h, 04.50.Cd. **ISSN 1870-9095**

I. INTRODUCTION

As known, mathematician Th. Kaluza in 1921 [1] proposed a model that seeks to unify the two fundamental forces, gravitation described by the Einstein's general relativity [2] and electromagnetism described by the Maxwell's equations [3] by multiplying gravity and the Maxwell stress tensor, where the product is a matrix in five dimensions, deriving an expression for the geodesics in the fifth dimension spacetime. The resulting equations can be separated into further sets of equations, yields the equivalent to Einstein field equations together with Maxwell's equations.

In 1926, Oskar Klein proposed that the extra spatial dimension is curled up in a circle of very small radius at each point in four-dimensional spacetime [4, 5]. This means the fifth dimension would have the topology of a circle, with a radius of the order of the Planck length. Five dimensional spacetime then has the topology $\boldsymbol{R}^4\times\boldsymbol{S}^1$, and the fifth coordinate y is periodic, $0 \leq \mathrm{m}\, y \leq 2\pi$, where m is the inverse radius of the circle. Thus, hypothetical pointlike particle moving a short distance along that axis would return to where it began completing a period. The distance that the pointlike particle can travel before reaching its initial position is said to be the size of the dimension. This spatial extra dimension is considered in the scale of very small particles, giving rise to the so-called Kaluza–Klein theory [6].

In this paper, gyroradius for a pointlike charged particle is derived from the Kaluza-Klein theory by introducing a magnetic field, where the hypothetic Klein's pointlike particle can be circumgyrating in a magnetic field to cover the volume that forms a determined particle in three dimensions. Some analogies of this derivation with the considered topology in the String Theory and M-Theory are commented.

II. KALUZA-KLEIN THEORY OVERVIEW

O. Klein expression for the period of a hypothetical pointlike charged particle in the fifth dimension spacetime is given by

$$l = \frac{hc\sqrt{2\kappa}}{q} = \frac{hc\sqrt{\frac{16\pi G}{c^4}}}{q} = \frac{h\sqrt{16\pi G}}{qc}, \tag{1}$$

where l is the length of the period, h is the Plank constant, c is the speed of light, q is the particle's electric charge and κ is the so-called Einstein gravitational constant, with G being Newton's constant.

Replacing with the known value for electron into expression (1), it is given that $l \approx 8\times10^{-33}$ m. Having the quantic moment expression defined as $p = h/l$, moment from expression (1) for a 5d spacetime reference gives

$$m\dot{x}_5 = p^5 = \frac{h}{l} = \frac{qc}{\sqrt{16\pi G}}, \tag{2}$$

where m is the rest mass of the pointlike particle and $\dot{x}_5 = v_5$ is the velocity of a hypothetical charged particle in the 5d spacetime.

III. INTRINSIC GYRORADIUS IN THE KALUZA-KLEIN THEORY

It is possible to confirm that expression (2) has not dimensional consistency, but gives the relation between gravity (given by G) and charge for a charged particle in a 5d spacetime. It is required to complement expression (2) in order to reach dimensional consistency [7], for instance by introducing terms of permeability and permittivity of free space and their equivalence with square of speed of light, hence

$$m\dot{x}_5 = \frac{q}{\sqrt{16\pi\mu_0\varepsilon_0^2 G}}, \tag{3}$$

where μ_0 is the permeability of free space and ε_0 is the permittivity of free space. Expression (3) conserves relation between gravity and charge as given in expression (2) having also dimensional consistency.

As considered in the Kaluza-Klein theory, fifth dimension would have the topology of a circle, where a pointlike charged particle defined in 5d spacetime describes a circular motion to cover a three dimensional reference through time. Thus, period of the circular motion of a pointlike charged particle is given by

$$T = 2\pi t = 2\pi\frac{m}{q\mathbf{B}} \therefore t = \frac{m}{q\mathbf{B}}, \tag{4}$$

where m is the rest mass of the particle, t is the time and $\mathbf{B}$ is the magnetic field. Considering velocity as distance per time and replacing in expression (4), hence

$$\dot{x}_5 = v_5 = \frac{r_5}{t} = \frac{r_5 q\mathbf{B}}{m}, \tag{5}$$

reordering terms of expression (3) to be compared with expression (5), yields

$$\dot{x}_5 = \frac{r_5 q\mathbf{B}}{m} = \frac{q}{\sqrt{16\pi\mu_0\varepsilon_0^2 G m}}. \tag{6}$$

Thus, magnetic field $\mathbf{B}$ is introduced in the complemented Klein's equation (3) by the analogy with period of a pointlike charged particle circumgyrating. Simplifying expression (6) and reordering, yields

$$r_5 = \frac{1}{\sqrt{16\pi\mu_0\varepsilon_0^2 G\mathbf{B}}}. \tag{7}$$

After the reduction of terms for charge and mass, expression (7) shows the relation between both, gravity and electromagnetism for a pointlike charged particle circumgyrating in a magnetic field at the scale of 5d spacetime. Verification of relation between gravity and electromagnetism as given in expression (7) can be confirmed by developing this expression to reach the Einstein field equation solution with magnetic stress tensor [8]. Thus, from expression (7), inverse of square of radius, yields

$$\frac{1}{r_5^2} = 16\pi G\mu_0\varepsilon_0^2\mathbf{B}^2, \tag{8}$$

$$\frac{1}{r_5^2} = 16\pi G\mu_0\varepsilon_0^2\left(\frac{qc}{4\pi R^2\varepsilon_0 c^2}\right)\mathbf{B} = \frac{16\pi G}{c^4}\left(\frac{qc\mathbf{B}}{4\pi R^2}\right),$$

where,

$$\mathbf{B} = \frac{\mu_0 qc}{4\pi R^2} = \frac{qc}{4\pi R^2\varepsilon_0 c^2}, \tag{9}$$

it is an equivalent expression of the magnetic field in terms of permittivity of free space.

A tensor is generally defined as stress. A stress field is generally a force per unit area. Thus, "magnetic stress tensor" can be defined in a simple way as the force (magnetic part of Lorentz force $\mathbf{F}_L$ for speed of light, in this case) per unit area A [9], where for a spherical surface is giving by

$$T^M_{\mu,\nu} = \frac{\mathbf{F}_L}{A_{\mu,\nu}} = \frac{qc\times\mathbf{B}}{4\pi R^2}, \tag{10}$$

where $T^M_{\mu,\nu}$ is the magnetic stress tensor and indexes μ,ν run 1, 2, 3; which is included in the expression (8). An equivalent expression that includes the electric field can be derived by developing expression (10) in terms of permeability and permittivity of free space, and reducing yields

$$T^M_{\mu,\nu} = \frac{qc\times\mathbf{B}}{4\pi R^2} = \frac{q\times\mathbf{B}}{4\pi R^2\mu_0\varepsilon_0 c} = \frac{1}{\mu_0 c}\mathbf{E}\times\mathbf{B}, \tag{11}$$

where $\mathbf{E}$ is the electric field and the Poynting vector [10] is given by

$$S_e = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}. \tag{12}$$

If the field is only magnetic some terms are reduced, expression (11) becomes

$$T^{M}_{\mu,\nu} = \frac{1}{\mu_0 c}\mathbf{E}\times\mathbf{B} = \frac{\mathbf{B}^2}{\mu_0}, \qquad (13)$$

where $\mathbf{B}^2 = B_x^{\ 2} + B_y^{\ 2} + B_z^{\ 2}$, which is a simplified equivalent expression of the magnetic Maxwell stress tensor [11], defined as

$$T^{M}_{\mu,\nu} = \frac{1}{\mu_0}\left(\mathbf{B}_\mu \mathbf{B}_\nu - \frac{\mathbf{B}^2}{2}\delta_{\mu,\nu}\right), \qquad (14)$$

where $\delta_{\mu,\nu}$ is Kronecker's delta, and it is proportional to the magnetic tension force [12], which is actually a pressure gradient and also a force density (N/m^3) that acts parallel to the magnetic field.

Furthermore, in two dimensions (for a given surface) scalar curvature is exactly twice the Gaussian curvature [13]. For an embedded surface in Euclidean space, this means that

$$S = \frac{2}{\rho_1 \rho_2}, \qquad (15)$$

where ρ_1, ρ_2 are the principal radii of the surface. For example, scalar curvature in S^3 of a sphere with radius r is equal to $2/r^2$.

In addition, from expression (1) it is given twice the Einstein gravitational constant in the Klein's solution given by expression (1), having

$$2\kappa = \frac{(2)8\pi G}{c^4}. \qquad (16)$$

Then, replacing expressions (14), (15) and (16) in expression (8), yields

$$2S_M = 2\left(\frac{2}{r_5^2}\right) = \frac{(2)8\pi G}{c^4} T^{M}_{\mu,\nu}. \qquad (17)$$

and simplifying, scalar curvature can be written as

$$S_M = \frac{2}{r_5^2} = \frac{8\pi G}{c^4} T^{M}_{\mu,\nu} = \frac{8\pi G}{c^4}\left(\frac{\mathbf{B}^2}{\mu_0}\right), \qquad (18)$$

that is the Einstein field equation solution with the magnetic stress tensor, verifying expression (7).

On the other hand, we can find out radius from expression (2) or its complemented expression (3) avoiding reduction of terms for charge and mass by considering that the pointlike charged particle in 5d spacetime is moving in a magnetic field **B**. Thus, dividing both sides of expression (3) by the magnetic field, hence

$$\frac{m\dot{x}_5}{\mathbf{B}} = \frac{mv_5}{\mathbf{B}} = \frac{q}{\sqrt{16\pi\mu_0\varepsilon_0^2 G}\mathbf{B}}, \qquad (19)$$

and reordering, yields

$$\frac{mv_5}{q\mathbf{B}} = \frac{1}{\sqrt{16\pi\mu_0\varepsilon_0^2 G}\mathbf{B}} = r_5, \qquad (20)$$

where $r_5 = r$ is the gyroradius (also known as radius of gyration, Larmor radius or cyclotron radius) [14], which is the radius of the circular motion of a charged particle in a magnetic field around a point called the guiding center, and it is proportional to the linear moment mv of the particle [15]. Expression (20) shows the intrinsic gyroradius in the Kaluza-Klein theory when a magnetic field is considered and terms for the dimensional consistency are included.

As known, when a charged particle moves through a magnetic field, it experiences a force defined by the Lorentz force given by the cross product of the velocity and magnetic field as

$$\mathbf{F}_L = q(\mathbf{v}\times\mathbf{B}), \qquad (21)$$

where **v** is the instantaneous velocity vector of particle.

Now if the velocity of the particle is perpendicular to the magnetic field, then Lorentz force will always act perpendicular to the direction of motion. Force provides the centripetal force and causing that particle moves in a circle (gyrate). Gyroradius or radius of the orbit can be derived from the magnetic centripetal force. According to the Newton's second law, it is given by

$$\mathbf{F} = ma_c = m\frac{v^2}{r} = qv_\perp \mathbf{B} \therefore r = \frac{mv_\perp}{q\mathbf{B}}, \qquad (22)$$

where a_c is the centripetal acceleration.

Let us now consider that initial velocity is not parallel or perpendicular to the magnetic field, but it forms a given angle α with respect the magnetic field. On one hand, motion parallel to the field is uniform while for perpendicular motion is circular. Combination of those two motions is a helical path through time. Radius of the helix is given by

$$r = \frac{mv}{q\mathbf{B}} = \frac{mv_0 \sin\alpha}{q\mathbf{B}}. \qquad (23)$$

Sense of the motion of the helix depends of the sign of the charge.

Frequency of this circular motion is known as the gyrofrequency or cyclotron frequency, defined in radian/second as

$$\omega = \frac{v_0}{r} = \frac{q\mathbf{B}}{m}. \qquad (24)$$

Those quantities only depend of the rate q/m and the magnetic field**,** and they are independent of velocity.
Then, charge of the formed pointlike particle (that we call q_1 with mass m_1) in 3d is equivalent to the charge density ρ_{q1} (charge per unit volume) created by the "small" pointlike charged particle in circular motion (that we call q_2 with mass m_2) in 5d spacetime when it covers a volume Vol_1 through time (Figure 1).

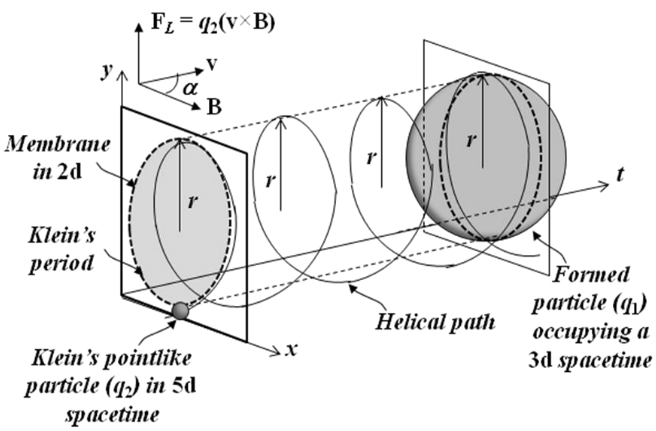

FIGURE 1. Charged particle q_2 moving in helical path through time in a magnetic field, covering volume of charged particle q_1.

Having that volume is equivalent to the matter per density of matter ρ_m, hence

$$q_1 \to \rho_{q1} = \frac{q_2}{Vol_1} = \frac{q_2\rho_m}{m_2} \therefore q_2 = \rho_{q1}Vol_1. \qquad (25)$$

The waveform described by the trajectory of the charged pointlike particle q_2 occupies a volume during its motion through time to form the particle q_1 (Figure 1) and its direction may be affected in presence of an electromagnetic field. Inducted magnetic orientation to the particle influences in its trajectory according to the vector of the magnetomotive force inducted to the particle by the magnetic field. The magnetic orientation (polarization) of the vibration frequency gives to the formed charged particle q_1 a defined gyromagnetic orientation (spin).

IV. ANALOGIES WITH THE STRING THEORY

It is noticed that this motion of pointlike particle has some analogies with the interpretation of topology for a particle in N spatial dimensions as described in the String theory (mainly with Type 1 [16]) and M-theory (mainly in the P-brane [17, 18, 19]), that is, a 0-brane is a zero-dimensional pointlike particle, a 1-brane is a string, that can either be open or closed, and a 2-brane is a "membrane" on a surface.

In this case, a considered pointlike particle in 5d spacetime (that is perceived as a pointlike particle of 0-brane from a 4d spacetime reference) describes a linear trajectory through time (worldline, Figure 2a), so that the particle does not cover a major space than its own volume. If the particle is now in motion along a one spatial dimension, vibrating with a given frequency, it will describe a linear path in 1d (1-brane, as a string) that will cover an area through time (worldsheet, Figure 2b). In this scenario, linear path by the vibration through time defines an open string [20] (since this string does not close in the time) that covers an area defining a corresponding topology with the String Type 1.

Now, if the particle is also in motion along an additional spatial dimension, vibrating with a given frequency, it will describe a curved path in 2d along a surface will be a membrane in 2d (2-brane, Figure 2c). Additional spatial dimension that the particle covers during its motion becomes from the progress of time, where vibration of the particle in two dimension space through time will cover an additional spatial dimension to conform a volume in three-dimensions (worldvolume, Figure 2c). It is an analogous scenario to the pointlike charged particle in circular motion in a magnetic field (Figure 1).

By applying the ideas of quantum mechanics to strings it is possible to deduce the different vibration modes of strings, and that each vibration state appears to be a different particle. Thus, mass of each particle and the characteristics with which it can interact, are determined by the way the string vibrates.

V. CONCLUSIONS

This analysis shows that the Kaluza-Klein theory intrinsically contains gyroradius effect for a hypothetical pointlike charged particle in 5d spacetime, which can be derived by introducing a magnetic field. This consideration is compatible with the Einstein field equation solution with the magnetic stress tensor. In this analogy, motion of this pointlike particle through the spatial dimension is like a vibration that covers those spaces, as described by the Kaluza-Klein theory, then forming a pointlike particle in three dimensions. It is noticed that this description has some analogies with the String theory and M-theory, where in this case, topology of the motion of the pointlike particle in 1d and 2d are considered to form a 1-brane as open string and a 2-brane, respectively.

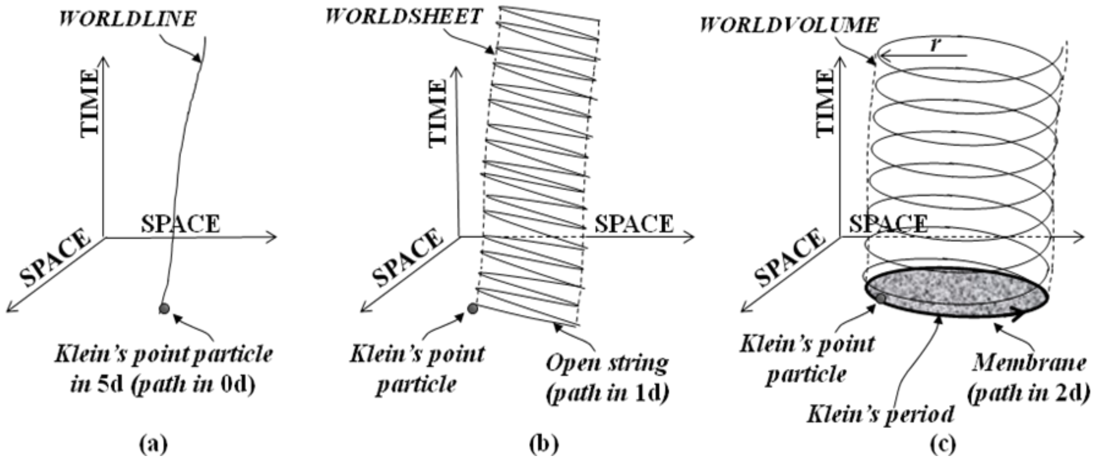

FIGURE 2.Trajectory of a particle in spacetime traces a worldline. Similarly, particle vibrating that of a string or a membrane sweeps out a worldsheet or worldvolume, respectively.

Regarding to the education, classical Kaluza-Klein theory is revisited describing the main concepts of this theory defined in the five-dimension space-time, where it is showed the possibility to apply some of the known equivalences to consider another possible properties from the classical theories.

ACKNOWLEDGEMENTS

The author would like to thank Professor Sergio S. Cornejo for the review and comments for this paper.

REFERENCES

[1] Kaluza, Th., *Zum Unitätsproblem in der Physik*, Sitzungsber. Preuss. Akad. Wiss. (Math. Phys., Berlin, 1921), pp. 966–972.

[2] Einstein, A., *The Foundation of the General Theory of Relativity*, Annalen der Physik **49**, 284–337 (1916).

[3] Maxwell, J. C., *A Treatise on Electricity and Magnetism*, Vol. II, 3rd Ed. (Oxford University Press, UK, 1904), pp. 178-179 and 189.

[4] Klein, O., *Quantentheorie und fünfdimensionale Relativitätstheorie*, Zeitschrift für Physik A **37**, 895–906 (1926).

[5] Klein, O., *The Atomicity of Electricity as a Quantum Theory Law*, Nature **118**, 516-516 (1926).

[6] Duff, M. J., *Kaluza-Klein Theory in Perspective*, In Lindström, Ulf (Ed.). *Proceedings of the Symposium 'The Oskar Klein Centenary',* (World Scientific, Singapore, 1994), pp. 22–35.

[7] Sandoval-Villalbazo, A., García-Perciante, A. L., García-Colín, L. S., *Hyperbolic heat equation in Kaluza's magnetohydrodynamics*, Gen. Rel. Grav. **39**,1287-1296 (2007) pp. 4-13 [arXiv:gr-qc/0601042].

[8] García-Duque, C. H. and García-Reyes, G., *General Magnetized Weyl Solutions: Disks and Motion of Charged Particles*, Gen. Relativ. Gravit. **43**, 11, 3001 (2011) [arXiv:1009.1084].

[9] Chen, W. and Han, D., *Plasticity for structural engineers*, (J. Ross Publishing, Florida, USA, 2007), pp. 46–71.

[10] Richter, F., Florian, M. and Henneberger, K., *Poynting's theorem and energy conservation in the propagation of light in bounded media*", Europhys, Lett. **84**, 67005 (2008), pp. 2-3 [arXiv:0710.0515].

[11] Bimonte G., Calloni, E. and Luigi Rosa, L., *The Gravity of Magnetic Stresses and Energy*, (Phys.Rev.D77:044026, 2008) [arXiv:0707.1077].

[12] Symon, K. R., *Mechanics*, 3rd Ed. (Addison-Wesley Publishing Company, Boston, Massachusetts, USA, 1971).

[13] Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry*, Vol. 1, (Interscience, New York, USA, 1963).

[14] Chen, F. F., *Introduction to Plasma Physics and Controlled Fusion*, Plasma Physics, 2nd Ed. Vol. 1 (Plenum Press, New York, USA, 1984), p. 208.

[15] Cornejo, A. G., *Perpendicularity between speed of light and time in the Kaluza-Klein Theory*, Lat. Am. J. Phys. Educ. **6**, 247–249 (2012).

[16] Gliozzi, F., Scherk, J. and Olive, D. I., *Supersymmetry, Supergravity Theories and the Dual Spinor Model*, Nucl. Phys. **B122**, 253 (1977).

[17] Lukas, A., Ovrut, B. A. and Waldram, D., *Five–Branes and Supersymmetry Breaking in M–Theory*, (JHEP 9904:009, 1999) [arXiv:hep-th/9901017].

[18] Lima, E., Ovrut, B. A. and Park, J., *Five-Brane Superpotentials in Heterotic M-Theory*, Nucl.Phys. B626 (2002) 113-164 [arXiv:hep-th/0102046].

[19] Duff, M. J., *The Theory Formely Known as Strings* (Scientific American, February 1998) pp. 64 - 67

[20] Angelantonj, C. and Sagnotti, A., *Open Strings*, Phys.Rept. **371**, 1-150 (2002*); Erratum-ibid*. **376**, 339-405 (2003), [arXiv:hep-th/0204089].