Fixing Bell-like from the covariance matrix approach

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Abstract
The covariance matrix is examined for a particular family of Bell-like states. It is found that both the trace and the determinant of the covariance matrix contain valuable information related directly to entanglement. By employing basic properties of entanglement, the family of Bell-like states are refined. Non trivial lower bounds to the determinant of the covariance matrix are found. These bound has to do with the so called quantum Fisher information and they are related to the degree of uncertainty on entanglement.

Keywords: Entanglement, covariance matrix, Bell-like states, quantum Fisher information.

Resumen
La matriz de covarianza es examinada para una familia particular de estados de tipo Bell. Se halla que tanto la traza como el determinante de la matriz de convarianza contiene información valiosa relacionada directamente al entrelazamiento. A través del empleo de las propiedades básicas de entrelazamiento, la familia de estados tipo Bell es refinada. Cotas inferiores al determinante de la matriz de covarianza son halladas. Estas cotas tienen que ver con la así llamada información cuántica de Fisher y ellas están relacionadas al grado de incertidumbre del entrelazamiento.

Palabras clave: Enredo, matriz de covarianza, estados cuánticos tipo campana, información cuántica de Fisher.

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I. INTRODUCCIÓN

Entanglement is a basic ingredient for quantum information, communication, and computation [1]. Related to this issue one immediate task becomes to find a criterion whether a given state is entangled or not [2]. In order to answer this question, the Peres-Horodecki inseparability criterion [3] or the so called *posivity under partial transpose* (PPT) might be a very useful criteria. Such a criteria provides a necessary and sufficient conditions for 2x2 and 2x3 dimensional system [3]. By the way, the PPT criterion is also valid for the case of infinite dimensional bipartite Continuous Variable (CV) states [4]. On the other hand, quantum correlated macroscopic atomic ensembles have attracted significant interest [2, 5]. For instance, experimental generation of entangled multiqubit states in trapped-ion systems, [6, 7, 8, 9] where individual particles can be manipulated, has been accomplished, opening new ways for scalable quantum information processing.

In the present work, we study the entanglement properties hidden in the covariance matrix for the case of Bell-like states $k_1|\uparrow_1\uparrow_2\rangle + k_2|\downarrow_1\downarrow_2\rangle$ and $\alpha_1|\uparrow_1\downarrow_2\rangle + \alpha_2|\downarrow_1\uparrow_2\rangle$. It is worth to point out that notable findings on the entanglement properties of the covariance matrix for the symmetric states $k_1|\uparrow_1\uparrow_2\rangle + k_2|\downarrow_1\downarrow_2\rangle$ have been done in Ref.[2]. Consequently we shall focus mainly on the states.

$$|\Psi\rangle = \alpha_1|\uparrow_1\downarrow_2\rangle + \alpha_2|\downarrow_1\uparrow_2\rangle$$
$$-1 < \alpha_1 \leq \alpha_2 \leq 1, \quad \alpha_1^2 + \alpha_2^2 = 1. \tag{1}$$

Such restrictions on the constants α_1 and α_2 imply that $\alpha_1 \leq \alpha_2 = \sqrt{1-\alpha_1^2}$ is equivalent to $-1/\sqrt{2} \leq \alpha_1 \leq 1/\sqrt{2}$.

In order to proceed further, we outline the basic tools for accomplishing our task. The most general two-qubits density can be expanded as follows [2]

$$\rho = \frac{1}{4}\left[I \otimes I + \sum_{i=x,y,z} \left(\sigma_{1i} s_{1i} + \sigma_{2i} s_{2i}\right) + \sum_{i,j=x,y,z} \sigma_{1i}\sigma_{2j} t_{ij} \right], \tag{2}$$

where I is the 2x2 unit matrix, $\alpha_{1i} = \alpha_i | \otimes I$ and $\sigma_{2i} = I \otimes \sigma_i$ being σ_x, σ_y, and σ_z the Pauli matrices, $s_{\alpha i} = \mathrm{Tr}\left(\rho\sigma_{\sigma i}\right)\left(\alpha = 1,2\right)$ denote average spin components of the α^{th} qubit and $t_{ij} = \mathrm{Tr}\left(\rho\sigma_{1i}\sigma_{2j}\right)$ are elements of the real 3 x 3 matrix T corresponding to the two-qubit correlations.

By using the decimal notation $|0\rangle = |\uparrow_1\uparrow_2\rangle$, $|1\rangle = |\uparrow_1\downarrow_2\rangle$, $|2\rangle = |\downarrow_1\uparrow_2\rangle$, and $|3\rangle = |\downarrow_1\downarrow_2\rangle$ then the state (1) takes the form $|\psi\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle$. Thus, the respective density matrix should be.

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha_1^2 & \alpha_1\alpha_2 & 0 \\ 0 & \alpha_1\alpha_2 & \alpha_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3)$$

By comparing Eqs. (2) and (3) we obtain

$$\rho 00 = \frac{1}{4}\left(1 + s_{1z} + s_{2z} + t_{zz}\right) = 0 \quad (4)$$

$$\rho 01 = \frac{1}{4}\left(s_{2x} + is_{2y} + t_{zx} + it_{zy}\right) = 0 \quad (5)$$

$$\rho 02 = \frac{1}{4}\left(s_{1x} + is_{1y} + t_{xz} + it_{yz}\right) = 0 \quad (6)$$

$$\rho 03 = \frac{1}{4}\left[t_{xx} - t_{yy} + i(t_{xy} + t_{yx})\right] = 0 \quad (7)$$

$$\rho 10 = \frac{1}{4}\left(s_{2x} - is_{2y} + t_{zx} - it_{zy}\right) = 0 \quad (8)$$

$$\rho 11 = \frac{1}{4}\left(1 + s_{1z} - s_{2z} - t_{zz}\right) = \alpha_1^2 \quad (9)$$

$$\rho 12 = \frac{1}{4}\left[t_{xx} + t_{yy} + i(t_{xy} - t_{yx})\right] = \alpha_1\alpha_2 \quad (10)$$

$$\rho 13 = \frac{1}{4}\left(s_{1x} + is_{1y} - t_{xz} - it_{yz}\right) = 0 \quad (11)$$

$$\rho 20 = \frac{1}{4}\left(s_{1x} - is_{1y} + t_{xz} - it_{yz}\right) = 0 \quad (12)$$

$$\rho 21 = \frac{1}{4}\left[t_{xx} + t_{yy} + i(t_{xy} - t_{yx})\right] = \alpha_1\alpha_2 \quad (13)$$

$$\rho 22 = \frac{1}{4}\left(1 - s_{1z} + s_{2z} - t_{zz}\right) = \alpha_2^2 \quad (14)$$

$$\rho 23 = \frac{1}{4}\left(-t_{zx} - it_{zy}\right) = 0 \quad (15)$$

$$\rho 30 = \frac{1}{4}\left[t_{xx} - t_{yy} - i(t_{xy} + t_{yx}) + t_{zx} - it_{zy}\right] = 0 \quad (16)$$

$$\rho 31 = \frac{1}{4}\left(s_{1x} + is_{1y} - t_{xz} - it_{yz} - t_{zz}\right) = 0 \quad (17)$$

$$\rho 32 = \frac{1}{4}\left(s_{2x} - is_{2y} - t_{zx} + it_{zy}\right) = 0 \quad (18)$$

$$\rho 33 = \frac{1}{4}\left(1 - s_{1z} - s_{2z} + t_{zz}\right) = 0. \quad (19)$$

As a result of the last equations we obtain

$$s_1 = \left(0, 0, 2\alpha_1^2 - 1\right), \quad (20)$$

$$s_2 = \left(0, 0, 1 - 2\alpha_1^2\right), \quad (21)$$

$$t = \begin{pmatrix} 2\alpha_1\alpha_2 & 0 & 0 \\ 0 & 2\alpha_1\alpha_2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (22)$$

As it can be observed from the above equation, the matrix t is diagonal.

The 3x3 covariance matrix can be written as [2]

$$v = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \quad (23)$$

Where

$$A_{ij} = \frac{1}{2}\left[\langle\{\sigma_{1i}, \sigma_{1j}\}\rangle - \langle\sigma_{1i}\rangle\langle\sigma_{1j}\rangle\right] = \delta_{ij} - \langle\sigma_{1i}\rangle\langle\sigma_{1j}\rangle = \delta_{ij} - s_{1i}s_{1j}, \quad (24)$$

$$B_{ij} = \frac{1}{2}\left[\langle\{\sigma_{2i}, \sigma_{2j}\}\rangle - \langle\sigma_{2i}\rangle\langle\sigma_{2j}\rangle\right] = \delta_{ij} - \langle\sigma_{2i}\rangle\langle\sigma_{2j}\rangle = \delta_{ij} - s_{2i}s_{2j}, \quad (25)$$

$$C_{ij} = \frac{1}{2}\left[\langle\sigma_{1i}, \sigma_{2j}\rangle - \langle\sigma_{1i}\rangle\langle\sigma_{2j}\rangle\right] = t_{ij} - s_{2i}s_{2j}. \quad (26)$$

The entanglement properties of the non separable states of the form given by Eq. (1) are contained in the matrix *C* of Eq. (26).

By using Eqs. (20) - (22) we obtain that

$$C = \begin{pmatrix} 2\alpha_1\alpha_2 & 0 & 0 \\ 0 & 2\alpha_1\alpha_2 & 0 \\ 0 & 0 & -1-\left(1-2\alpha_1\right)^2 \end{pmatrix}. \quad (27)$$

From the above equation the trace of the covariance matrix is

$$\mathrm{Tr}\, C = 4\alpha_1\alpha_2 - 1 - \left(1 - 2\alpha_1\right)^2. \quad (28)$$

In the Figure 1 we plot the above quantity as a function of α_1 in the range $-1/\sqrt{2} \le \alpha_1 \le 1/\sqrt{2}$.

On the other hand, in Figure 2 we have plotted the following mathemathical expression of the determinant of the covariance matrix as given by Eq. (27)

$$\mathrm{Det}\, C = -4\alpha_1^2\alpha_2^2\left[1 + \left(1 - 2\alpha_1\right)^2\right] \quad (29)$$

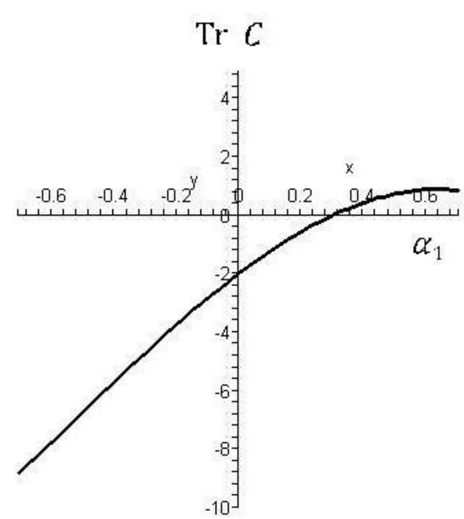

FIGURA 1. Trace of the covariance matrix *C* as given by Eq. (28) as a function of α_1 in the range $-1/\sqrt{2} \le \alpha_1 \le 1/\sqrt{2}$.

From the Figure 1, the following conclusions can be extracted: (i) The trace of the covariance matrix C is monotone and increasing; (ii) The trace of C is the sum of its eigenvalues, in order that this represents entanglement, this must be positive definite; (iv) the above constrains further the range of values of α_1 to 0.3 $\leq \alpha_1 \leq 0.72$; (iv) The trace of C takes its maximal value equal to one for the maximally entangled Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$; (v) The trace of C takes its minimal value for the state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$; (vi) Only in the range $0.3 \leq \alpha_1 \leq 0.72\left(\alpha_2 = \sqrt{1-\alpha_1^2}\right)$.

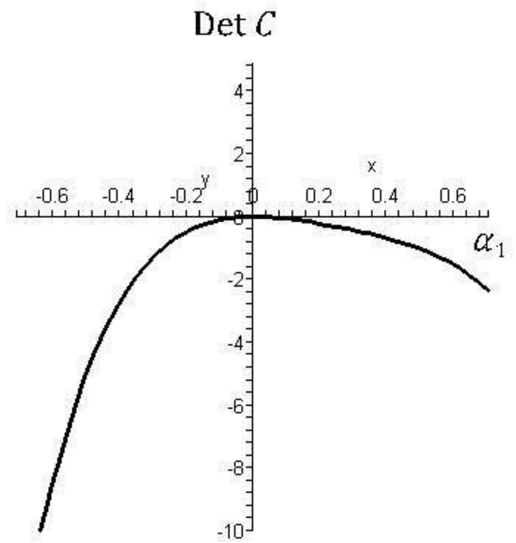

FIGURE 2. Determinant of the covariance matrix C as given by Eq. (28) as a function of α_1 in the range $-1/\sqrt{2} \leq \alpha_1 \leq 1/\sqrt{2}$.

The Bell-like state $|\Psi\rangle = \alpha_1 |\uparrow_1\downarrow_2\rangle + \alpha_2 |\downarrow_1\uparrow_2\rangle$ produce a positive definite trace. On the other hand, from the Figure 2 one can conclude the following: (a) The value of the determinant of C is always negative; (b) Due that this determinant is the product of the eigenvalues of C, then figure 2 is consistent with Figure 1 in the region $-1/\sqrt{2} \leq \alpha_1 \leq 0$ (i.e. where both are negative); (c) The sign of the slope of give us information about the characteristics of the Bell state, the slope is positive for Bell-like states of the form $|\Psi^-\rangle$ while for negative slopes, the value of α_1 and α_2 are both positives; (d) The magnitude of the slopes of the determinant of C are maximal for the Bell states $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

The covariance matrix has been examined for the particular family of states given by Eq. (1). It is found that this matrix contains valuable information related directly to entanglement. Such information is contained mainly through its trace. From the basic properties of entanglement stringent bounds to the family of states of Eq. (1) are imposed. Non trivial lower bounds to the determinant of the covariance matrix are found. These bounds have to do with the so called quantum Fisher information [10] and they are related to the degree of uncertainty of the entanglement.

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