A counterexample to Boyer-Plebañski conjecture

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Resumen

We show that the space-time obtained by Pandey –Sharma and Modak is a counterexample for the Boyer-Plebañski conjecture about 4-spaces with spherical symmetry and conformally flat.

Keywords: 4-space of class one, Boyer-Plebañski proposal.

Resumen

Se demuestra que el espacio-tiempo obtenido por Pandey-Sharma y Modak es un contraejemplo para la conjetura de Boyer-Plebañski sobre 4-espacios conformalmente planos con simetría esférica.

Palabras clave: 4-espacio de clase uno, propuesta Boyer-Plebañski

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Pandey-Sharma [1] and Modak [2] considered the space-time with spherical symmetry:

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 sin^2\theta dr^2 - (1+B(t)r^2)^2 dt^2\,, \quad (1)$$

and $B(t)$ an arbitrary function. In Synge [3], we find useful formulae to determine the non-null components of Riemann, Ricci and Einstein tensors, and the scalar curvature for (1)

$$\left[\left(x^k\right)=\left(r,\theta,\varphi,t\right)\right]:$$

$$R_{1414} = 2B\left(1+Br^2\right), R_{3434} = sin^2\theta R_{2424} = r^2 sin^2\theta R_{1414},$$

$$R_{11} = \frac{2B}{1+Br^2}, \quad R_{33} = sin^2\theta R_{22} = r^2 sin^2\theta R_{11},$$

$$R_{44} = -6B\left(1+Br^2\right),\ R = \frac{12B}{1+Br^2},\ G_{11} = -\frac{R}{3},$$

$$G_{33} = sin^2\theta\, G_{22} = r^2 sin^2\theta\, G_{11}, \quad G_{44} = 0, \quad (2)$$

and the corresponding Weyl tensor is equals to zero, thus (1) is conformally flat.

On the other hand, we have the Boyer-Plebañski conjecture [4]:

"*If R_4 is conformally flat with spherical symmetry, then it has class one*", (3)

therefore (1) should be of class one, however, we can prove that this false. In fact, if (1) has class one then it is necessary the existence of the second fundamental form $b_{ij}=b_{ji}$ verifying the Gauss equation:

$$R_{ijkm} = \varepsilon\left(b_{ik}b_{jm} - b_{im}b_{jk}\right),\ \varepsilon = \pm 1. \quad (4)$$

that is [5],

$$pb_{ac} = \frac{k_2}{48} - \frac{1}{2} R_{ajkc}\ G^{jk}\,, \quad (5.a)$$

where

$$p^2 = -\frac{\varepsilon}{6}\left(\frac{R}{24}k_2 + R_{imnq}\, \overline{G^{iq}}\, G^{mn}\right) \gtrless 0,$$

$$k_2 = 4\, R^{ij} R_{ij} - R^2 - R^{ijmn}\ R_{ijmn} \quad (5.b)$$

If we employ (2) into (5.b) results that $K_2=p=0$, and (5.a) gives the condition:

$$R_{ajkc}\, G^{jk} = 0, \quad (6)$$

but it is easy to see that $R_{4jk4}\, G^{jk} = 24\, B \neq 0$ [because $B = 0$ would imply that (1) is flat]. Thus, (6) is incorrect for (1) and therefore this space- time of Pandey-Sharma and Modak, conformally flat with spherical symmetry, has not class one; then it is a counterexample for the conjecture (3) of Boyer-Plebañski.

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