Heat engine with finite thermal reservoirs and nonideal efficiency

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Abstract
The performance of an irreversible heat engine operating between two thermal reservoirs with finite, temperature-independent heat capacity is analyzed. For this purpose, a dynamic second-law efficiency is introduced and assumed to be constant. As the first-law efficiency increases from zero up to the Carnot limit, the common final temperature of the reservoirs interpolates between the arithmetic and geometric mean of their initial temperatures. The total output work and entropy change of the reservoirs are computed and related to the static efficiencies. The dynamic and static efficiencies are shown to be approximately equal to each other when the temperature of the cold reservoir is at least 10% of the temperature of the hot reservoir.

Keywords: Heat engine, thermodynamic efficiencies, work and heat.

Resumen
Se analiza el desempeño de un motor térmico funcionando en un proceso irreversible entre dos depósitos térmicos finitos, con capacidad térmica independiente de la temperatura. Para este fin, se introduce una eficiencia dinámica de la segunda ley que se supone constante. Como la eficiencia de la primera-ley aumenta de cero hasta el límite de Carnot, la temperatura final común de los depósitos se interpola entre la media aritmética y la media geométrica de sus temperaturas iniciales. La producción total de trabajo y el cambio de entropía de los depósitos se calcula y se relaciona con las eficiencias estáticas. Se muestra que las eficiencias dinámica y estática son aproximadamente iguales entre sí cuando la temperatura del depósito frío es al menos el 10% de la temperatura del depósito caliente.

Palabras clave: Motor térmico, eficiencia termodinámica, calor y trabajo.

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I. INTRODUCTION

A heat engine outputs work W (over a time interval that is long compared to the period of a single cycle) by extracting heat $Q_{\rm H}$ from a hot thermal reservoir and dumping heat $Q_{\rm C}$ into a cold reservoir. According to the first law of thermodynamics, the relation between these three energy transfers is $W = Q_{\rm H} - Q_{\rm C}$, and one can correspondingly define a first-law efficiency for the engine as $\varepsilon \equiv W/Q_{\rm H} = 1 - Q_{\rm C}/Q_{\rm H}$. An infinite thermal reservoir is defined to be one that can absorb or donate heat without change in its temperature. Assuming the hot and cold reservoirs are infinite, with constant temperatures of $T_{\rm H}$ and $T_{\rm C}$, respectively, then it is a standard topic in an introductory thermodynamics course to show that the first-law efficiency cannot exceed the Carnot limiting value of $\varepsilon_{\rm c} = 1 - T_{\rm C}/T_{\rm H}$. This result follows from the second law of thermodynamics and consequently some authors [1] have introduced a second-law efficiency as $\eta \equiv \varepsilon/\varepsilon_{\rm c}$. An advantage of this definition is that η can vary over the entire range from 0% (in which case no work is extracted, a situation that could be described as "maximally irreversible" operation of the engine) up to 100% (when the greatest possible amount of work is output, corresponding to an engine running reversibly), unlike ε whose upper limit depends on the temperatures of the reservoirs used.

An interesting variation [2, 3] on a heat engine consists in computing the total output work W if the thermal reservoirs are finite (rather than infinite) with specified heat capacities (taken for simplicity to be independent of temperature over the range from $T_{\rm C}$ to $T_{\rm H}$), when the engine is driven to exhaustion (i.e., until the two reservoirs reach a common final temperature T). A number of textbooks [4] have left this calculation as an end-of-chapter problem where both reservoirs have the same heat capacity C. In all of these references, it is assumed that the engine operates reversibly during its entire course of operation. This assumption does not imply that its first-law efficiency $\varepsilon_{\rm c}$ remains constant because that Carnot limit decreases to zero as $T_{\rm C}$ and $T_{\rm H}$ approach each other in value. Instead, it is η that remains constant, namely at a value of 100%. In the present paper, a more challenging version of the problem is analyzed in which η remains constant at *any* value between 0% and 100%. This model more realistically describes an engine having finite fuel and whose performance is less than ideal.

II. ANALYSIS

For a real engine, the rates of energy transfer and of changes in the reservoir temperatures vary from one phase of the cycle to another, say from an adiabatic compression to an isothermal exhaust step. This detailed variation can be ignored by averaging the transfers and changes over an entire cycle, assuming the time to reach fuel exhaustion is long compared to the period of one cycle. Infinitesimal work, heat, and temperature changes in these average values will then be denoted by dW, dQ_C, dQ_H, dT_C, and dT_H. The usual caveat applies that dW and dQ are not exact differentials of state functions. The standard "heat engine" sign conventions will be adopted in which dW and dQ are always positive; it is to be implicitly understood that dQ_H represents heat transfer *out of* the hot reservoir, dQ_C represents heat transfer *into* the cold reservoir, and dW represents work *output* by the engine. On the other hand, temperature is a function of state and will be computed by integration. It would therefore be confusing to insist that dT must be positive. As the engine runs, dT_H is negative because the hot reservoir is decreasing in temperature from an initial value of T_{H0} to the final value T. Meanwhile $dT_C > 0$ as the cold reservoir warms up from an initial value of T_{C0} to the same final value T. Taking both reservoirs to have a temperature-independent heat capacity C, it then follows that

$$dQ_H = -CdT_H \quad \text{and} \quad dQ_C = CdT_C \,. \tag{1}$$

At any point during the operation of the engine, its dynamic[1] first-law efficiency can be defined as

$$\varepsilon \equiv \frac{dW}{dQ_H} = 1 - \frac{dQ_C}{dQ_H} = 1 + \frac{dT_C}{dT_H} \tag{2}$$

using Eq. (1) in the last step. The corresponding Carnot limiting value is

$$\varepsilon_c = 1 - \frac{T_C}{T_H} \tag{3}$$

where T_C and T_H are the reservoir temperatures at that instant. Although both of these first-law efficiencies decrease with time, their ratio (defining the dynamic second-law efficiency η) is assumed to be constant. Rearranging terms then leads to a differential equation for T_C as a function of T_H,

$$T_H \frac{dT_C}{dT_H} + \eta T_C = (\eta - 1) T_H \,, \tag{4}$$

which is first order, inhomogeneous, and linear with variable coefficients. Noting that the homogeneous equation is of the power-law form, the complementary solution is $AT_H^{-\eta}$ where A is a constant that will be fit to the initial conditions. Furthermore, by trying a particular solution of the inhomogeneous equation that is proportional to T_H, one obtains $(\eta - 1) T_H / (\eta + 1)$. Adding together these complementary and particular solutions and fitting to the initial temperatures of the two reservoirs gives

$$\frac{T_C}{T_{H0}} = \left(\frac{T_{H0}}{T_H}\right)^{\eta} \left\{ \frac{T_{C0}}{T_{H0}} + \left(\frac{1-\eta}{1+\eta}\right) \left[1 - \left(\frac{T_H}{T_{H0}}\right)^{1+\eta} \right] \right\}. \tag{5}$$

Next, set $T_C = T_H = T$ to find the common final temperature of the two reservoirs in dimensionless form,

$$\frac{T}{T_{H0}} = \left[\frac{(T_{C0} + T_{H0}) + \eta (T_{C0} - T_{H0})}{2 T_{H0}} \right]^{1/(1+\eta)} . \tag{6}$$

This final temperature decreases monotonically with increasing second-law efficiency for fixed initial reservoir temperatures. Specifically $T = T_{min}$ when $\eta = 1$, and $T = T_{max}$ when $\eta = 0$, where

$$T_{min} = \sqrt{T_{C0} T_{H0}} \tag{7}$$

is the geometric average of the initial temperatures, and

$$T_{max} = \frac{T_{C0} + T_{H0}}{2} \tag{8}$$

is their arithmetic average. For example, if $T_{C0} = 100$ K and $T_{H0} = 900$ K, then $T_{min} = 300$ K and $T_{max} = 500$ K. It is convenient to use these limiting temperatures to express the common final temperature in normalized form as

$$\tilde{T} \equiv \frac{T - T_{min}}{T_{max} - T_{min}} \tag{9}$$

which according to Eqs. (6) to (8) is a function of η and of the initial temperature ratio $t \equiv T_{C0}/T_{H0}$. Like the second-law efficiency, $\tilde{T}$ has been defined such that it can theoretically vary between 0 (in which case $T = T_{min}$) and 1 (when $T = T_{max}$). Similarly t has a lower limit of 0 (when T_{C0} approaches absolute zero temperature) and an upper limit of 1 (when T_{C0} is only infinitesimally smaller in value than T_{H0}). In Fig. 1, $\tilde{T}$ is plotted versus η for various values of t between 0 and 1. In particular, l'Hôpital's rule can be used to prove that $\tilde{T} \to 1 - \eta$ as $t \to 1$.

[1]In analogy to the distinction between dynamic resistance (dV / dI for voltage V across and current I through a diode say) and static resistance (V / I), a distinction can be made between *dynamic* efficiencies (given by the incremental energy transfers at some instant) and *static* efficiencies (determined by the total heat and work transferred over the entire duration of the engine's operation).

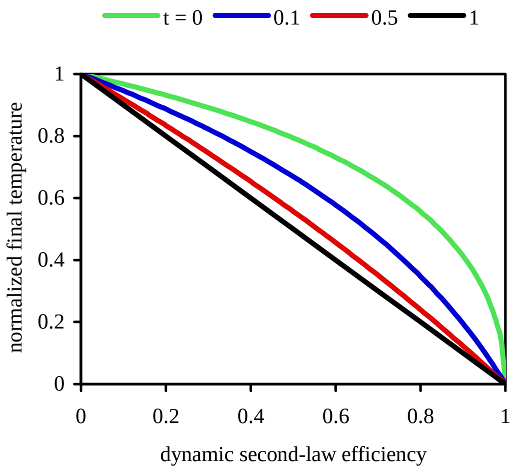

FIGURE 1. Plot of $\tilde{T}$ versus η for the values of *t* indicated in the legend.

The total work output by the engine is the difference in heat transfers for the two reservoirs, which can be related in turn to their net temperature changes,

$$W = Q_H - Q_C = C(T_{H0} - T) - C(T - T_{C0}). \quad (10)$$

Using Eqs. (8) and (9), this result can be rewritten as

$$\frac{W}{W_{max}} = 1 - \tilde{T} \quad \text{where} \quad W_{max} \equiv 2C(T_{max} - T_{min}). \quad (11)$$

The maximum work that can be extracted is W_{max}, equal to the product of the combined heat capacity (2*C*) of the two reservoirs and the maximum difference in the final temperatures. The maximum work is achieved for reversible operation of the engine when $\eta = 1$ so that $\tilde{T} = 0$. On the other hand, zero work is output for maximally irreversible operation of the engine when $\eta = 0$. The final temperature of the reservoirs is then the same as if one simply placed them in direct thermal contact with each other—the engine acts like a perfectly conducting thermal channel between them. To further substantiate this picture, the total entropy change of the reservoirs can be calculated as

$$\Delta S \equiv \Delta S_H + \Delta S_C = -C\ln\left(\frac{T_{H0}}{T}\right) + C\ln\left(\frac{T}{T_{C0}}\right) = 2C\ln\left(\frac{T}{T_{min}}\right). \quad (12)$$

This entropy change is zero for the reversible case $\eta = 1$, and it is maximum when $\eta = 0$.

Finally, the total work and heat transfers can be used to define *static* first- and second-law efficiencies as $\varepsilon^* \equiv W / Q_H$ and $\eta^* \equiv \varepsilon^* / \varepsilon_c^*$, respectively, where $\varepsilon_c^* = W_{max} / C(T_{H0} - T_{min})$ is the maximum possible static efficiency (for given initial reservoir temperatures), achieved if the engine always runs at the Carnot limit. Using Eqs. (7) and (10), one can prove that

$$\varepsilon_c^* = 1 - \sqrt{\frac{T_{C0}}{T_{H0}}}. \quad (13)$$

Leff [2] has pointed out the remarkable similarity of this result to the Curzon-Ahlborn efficiency of an endoreversible engine optimized for maximum power output [5, 6]. The static second-law efficiency is plotted in Fig. 2 against the dynamic second-law efficiency for the same values of *t* as in Fig. 1. Notice that $\eta^* \approx \eta$ for $t \gtrsim 0.1$, the same range of values of the initial temperature ratio as discussed by Leff. The two second-law efficiencies can be shown to be exactly equal to each other in the limit as $T_{C0} \to T_{H0}$ using l'Hôpital's rule.

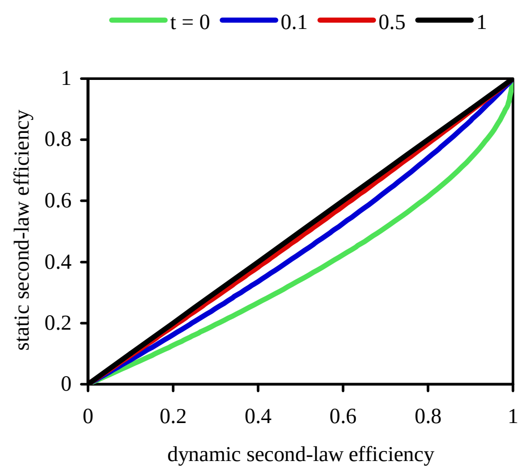

FIGURE 2. Plot of η^* versus η for the values of *t* indicated in the legend.

III. CONCLUSIONS

Previous results on the Carnot performance of heat engines using thermal reservoirs of finite heat capacity have been extended to more realistic, irreversible operation. Distinctions between dynamic (differential) and static (total) efficiencies on the one hand, and between first-law and second-law efficiencies on the other hand, have been introduced. The final reservoir temperatures, total output work, and net entropy change of the reservoirs have been computed. Calculation and graphing of many of these results (with suitable guidance) would make good homework problems in an undergraduate thermodynamics course.

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