Linear relationships in heat transfer

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Abstract

The use of linear relationships that can appear in heat transfer phenomena is described using a simple physical experimental situation in which the temperature evolution with time in a sample heated with low intensity continuous light is measured. These questions should be included in the introductory physics curricula of science and engineering studies to teach aspects from different branches of physics (for example thermodynamics) and mathematics (ranging from functional analysis to differential equations).

Keywords: Conduction heat transfer, Radiation, Convection.

Resumen

El uso de relaciones lineares que pueden aparecer en fenómenos de transferencia de calor es descrito utilizando una situación experimental sencilla en la cual la evolución de la temperatura con el tiempo es medida en una muestra calentada con luz de poca intensidad. Estas cuestiones podrían ser incluidas en el curriculum de Física de carreras de Ciencias e Ingeniería para enseñar aspectos de diferentes ramas de la Física (por ejemplo de Termodinámica) y de Matemática (desde análisis funcional hasta ecuaciones diferenciales).

Palabras clave: Transferencia de calor conducción, Radiación, Convección.

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There are several kinds of mathematical functional relationships in science between changing entities called variables, such as the exponential, the parabolic, the trigonometric, etc. One of the most common of them is the linear relationship, where an incremental change in one variable is matched by a proportional variation in the other. We can find several variables that depends linearly to one another, among others the velocity of a body and its displacement in which the former is constant, the applied voltage and the electrical resistance in a metal, the mass and the density of an object, the vapour pressure of a substance and its temperature, the gravitational and the electrostatic force between two charged objects and the inverse of the square of the distance between them, and so on. Linear equations can be written in the form of $y=mx+b$, in which x is the independent variable, y is the dependent variable, m is the slope, and b is the y-intercept. These equations appear to be straight lines in a xy-coordinate graph. Often the use of the logarithmic function allows the linearization of mathematical equations: It is well known that the exponential and the potential functions get linear in a semi-log and in the log-log plot of the dependent versus the independent variable respectively, a fact that is very often used in data processing to obtain typical parameters characterizing these functions. In this way we find, for example, that the logarithm of the Molar concentration versus time is a linear graph as well as the plot of the logarithm of the electrostatic force between two charged objects versus the logarithm of the distance between them.

But not always it is possible to find a linear relationship between the variables involved in a given problem, what makes sometimes difficult its solution, which must be found often numerically. Although the existence of powerful computational methods allows one today with relative ease to hand non-linear equations, a better physical insight in a studied problem can be obtained by means of analytical expressions, where particular limiting and asymptotic cases could be analyzed and, at the same time, could be easier programmed than complicated equations. Then, the use of linear relationships is always advantageous.

Here we will present a typical situation that can be encounter in thermal physics experiments, whose interpretation can involves non-linear equations. We will shown how a carefully analysis of the problem allow one to find the conditions for which these expressions become linear, what can make easier the look for an analytical solution of the problem.

Consider that a thin slab of a solid sample of thickness L is heated which a light beam that is uniformly focused onto one of its surfaces. On the opposite side, its temperature can be monitored as a function of time, for example with a thermocouple. The variation with time, t, of the heat generated in the sample, Q, due to the absorption of light of incident power P_i (W), is given by [1]

$$\partial Q / \partial t = P_i - q \ , \tag{1}$$

where q is a term taking into account the power losses by radiation, convection and conduction.

It is well known that any temperature difference within a physical system causes a transfer of heat from the region of higher temperature to the one of lower. This transport process takes place until the system has become uniform temperature throughout. Thus, the parameter q should be some function of the temperatures, T_1 and T_2, of both the regions involved (we will suppose that $T_2 > T_1$). It is denoted as the heat flux (units of W) and its form depends on the nature of the transport mechanism, which can be one of the three mentioned above or a coupling of them. The dependence of the heat flux on the temperature is in general non linear, a fact that makes quite difficult calculations using the energy balance equation (1).

On the one hand we see that for radiation, *i.e.* the continuous energy interchange between separated bodies by means of electromagnetic waves, the net rate of heat flow, q_{rad}, radiated by a body surrounded by a medium at a temperature T_1 is given by the Stefan-Boltzmann Law [2]

$$q_{rad} = \sigma A \varepsilon \left(T_2^{\ 4} - T_1^4 \right) \ , \tag{2}$$

where σ is the Stefan-Boltzmann constant, A is the surface area of the radiating object and ε is the total emissivity of its surface having absolute temperature T_2.

On the other hand, it is a well known fact that heat convection takes place by means of macroscopic fluid motion. It can be caused by an external source (forced convection) or by temperature dependent density variations in the fluid (free or natural convection). In general, the mathematical analysis of convective heat transfer is extremely complex [3]. Often problems can be solved only numerically or graphically. But convective heat flow, in its most simple form, i.e. heat transfer from surface of wetted area A and temperature T_2, to a fluid with a temperature T_1 , for small temperature differences $\Delta T = T_2 - T_1$ is given by the (linear with temperature) Newton's law of cooling,

$$q_{conv} = h_{\mathrm{conv}} A \left(T_2 - T_1 \right) = h_{\mathrm{conv}} A \Delta T \ . \tag{3}$$

The convective heat transfer coefficient, h_{conv} ($\mathrm{Wm^{-2}K^{-1}}$), is a variable function of several parameters of different kinds but independent on ΔT.

The third mechanism, called thermal conduction, can be understood in a simple way as a microscopic down-temperature diffusion process of heat within solids and stagnant fluids. The local heat flow-rate in some direction, r, is governed by Fourier's Law [2]

$$q_{cond} = -kA \nabla T \ . \tag{4}$$

The thermal conductivity, k (W/cmK), is expressed as the quantity of heat transmitted per unit time, t, per unit area, A, and per unit temperature gradient $\nabla T = \partial T / \partial \mathbf{r}$.

Now we are in condition to get back to Eq. (1). If we want to calculate the rise of temperature, ΔT, of the back sample's surface we must express the heat term of Eq. (1) as a function of that increase. It is given by the relationship

$$Q = \rho c V \Delta T \ , \tag{5}$$

where ρ is the density, c is the specific heat and $V = AL$ is the sample's volume. Differentiation of Eq. (5) with respect to time and substitution into Eq. (1) leads to:

$$\frac{\partial \Delta T}{\partial t} + \frac{q}{\rho c V} - \frac{P_i}{\rho c V} = 0 \ , \tag{6}$$

where q is specified by the sum of the radiation, convection and conduction terms given by Eq. (2), (3) and (4) respectively. But we can see that, whereas sufficiently small convective rates of heat flow can be considered as linearly dependent on temperature difference (see Eq. (3)), the radiation and conduction dependence on the temperature are described by non-linear relationships - Eqs. (2) and (4) respectively. This non-linearity makes complicated the analytical solution of the energy conservation law as given by Eq. (6).

But fortunately in the most often situations that we can find in daily live and professional practice the involved temperature differences are sufficient small so that, as we will shown later, the solution of Eq. (6) can be obtained in a straightforward way. Typical examples can be found when the sun rays irradiate our bodies, in optical experiments involving low intensity laser beams, in dynamic therapies where tissues are heated using mostly infrared radiation, among others. In these examples the temperature increases due to light absorption followed by light into heat energy conversion are much smaller than the ambient room temperature[1].

A glance at Eq. (2) shows that if the temperature difference $\Delta T = T_2 - T_1$ is small, then one should expand it as Taylor series around T_1 obtaining a linear relationship:

$$q_{\mathrm{rad}} = 4 \sigma A \varepsilon T_1^3 \left(T_2 - T_1 \right) = h_{rad} A \Delta T \tag{7}$$

If we compare this equation with Eq. (3) we can conclude that in this case $h_{\mathrm{rad}} = 4 \sigma \varepsilon T_1^{\ 3}$ can be considered as a radiation heat transfer coefficient.

On the other hand, for one-dimensional steady state conduction in extended samples of homogeneous and

[1] While absolute temperatures have different values when expressed in different units, temperature differences are always the same. For example, in the discussed physical situation, the absolute temperature, T_0, of an object at 300 K (27 ^{0}C) must be compared with the temperature increase of the same object above T_o of, say, ΔT=20 K = 20 ^{0}C.

isotropic materials and for small temperature gradients, Fourier's law can be integrated in each direction to its potential form. In rectangular coordinates it reads[2]:

$$q_{cond} = kA\frac{T_2 - T_1}{x_2 - x_1} = \frac{kA\Delta T}{\Delta x} = \frac{\Delta T}{R_T} = h_{cond} A\Delta T\ . \quad (8)$$

Here T_1 and T_2 represent two planar isotherms at positions x_1 and x_2, respectively. Using an analogy with electrical conduction, and introducing the concept of thermal resistance, R_T, Eq. (8) is often denoted as Ohm's law for thermal conduction [2]. Comparing with Eq. (3) we see that the parameter h_{cond} has been incorporated in Eq. (8) as the conduction heat transfer coefficient.

Then, substituting Eqs. (3), (7) and (8) into Eq. (6) leads to:

$$\frac{\partial \Delta T}{\partial t} + \frac{h}{\rho c L}\Delta T - \frac{P_i}{\rho c V} = 0\,. \quad (9)$$

A linear ordinary differential equation with constant coefficients, where the overall heat transfer coefficient is given by:

$$h = h_{conv} + h_{cond} + h_{rad}\,. \quad (10)$$

The solution of Eq. (9) is well known [5]:

$$\Delta T_{\uparrow} = \frac{P_i}{Ah}\left(1 - \exp(-\frac{t}{\tau})\right). \quad (11)$$

The parameter

$$\tau = \frac{L\rho c}{2h}\ , \quad (12)$$

is often called the relaxation time.

These results have been used before in a technique known as the temperature relaxation method under continuous illumination for the measurement of the specific heat capacity (ρc) of thin, small solid samples of known thicknesses [4]. It can be calculated from the τ value which can be obtained by the least squares fit of experimental curves ΔT versus t to equation (11) in the range of measurement times in which the exponential behavior is observed. This range can be determined using semi-log plots in order to avoid uncertainties due to deviations from the theoretical model.

In the original variants the sample is supported adiabatically using a poor heat conductor in a reservoir where vacuum is performed in order to neglect heat losses by conduction and convection, so that the linear relationship given by Eq. (7) can be used in a straightforward manner to interpret the results. Experiments analyzing the influence of convection on the results have been also reported recently [5]. To the best of the author knowledge the influence of conduction have been only discussed in the past by solving the partial differential heat diffusion equation with the boundary conditions describing the physical problem [6].

In summary, the phenomenological aspects described here suggest the possibility of dealing with them in advanced or introductory physics or engineering courses. Although non-linear phenomena and relationships are very often in physics this article shows one example on how linearization of physical formulas can be very useful in practice and hopefully it will aid to suggest teachers extend this theme to a wider spectrum of phenomena.

On the other hand, the here described experimental situation can be also helpful to teach technical questions related to the use of graphical methods such as semi-log plots to handle data, as well as the physics related to the problem, that involves aspects of several branches such as heat, thermodynamics, optics, and of mathematics, namely ordinary differential equations, exponential functions, etc.

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[2] We show for simplicity only the absolute value of $q_{const.}$.